Lecture 15 Addition of angular momentum

1. $\vec{J} = \vec{J}_1 + \vec{J}_2$ remains angular momentum

2. $j = j_1 + j_2,\ j_1 + j_2 - 1,\ \cdots\ |j_1 - j_2|$

3. $$|j_1 j_2 ; jm\rangle = \sum_{m_1 m_2} |j_1 m_1\rangle |j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | jm\rangle$$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{jm} |j_1 j_2 ; jm\rangle \langle jm | j_1 m_1 j_2 m_2\rangle$$

§ Angular momentum addition

$[J_{1i}, J_{1j}] = i\epsilon_{ijk} J_{1k}, \quad [J_{2i}, J_{2j}] = i\epsilon_{ijk} J_{2k}$

then $\vec{J} = \vec{J}_1 + \vec{J}_2$ remains an angular momentum

$[J_i, J_j] = i\epsilon_{ijk} J_k$. Nevertheless $\vec{J}_1 - \vec{J}_2$ is no longer an angular momentum.

Consider two sets of eigenstates $|j_1 m_1\rangle$ for J_1^2, J_{1z}, and $|j_2 m_2\rangle$ for J_2^2, J_{2z}. The direct product $|j_1 m_1 j_2 m_2\rangle$

$$J_{1,2}^2 |j_1 m_1\rangle|j_2 m_2\rangle = j_{1,2}(j_{1,2}+1) |j_1 m_1\rangle|j_2 m_2\rangle,$$

$$J_{1,2,z} |j_1 m_1\rangle|j_2 m_2\rangle = m_{1,2} |j_1 m_1\rangle|j_2 m_2\rangle.$$

Then we would like to superpose them to form the eigenstates $|jm\rangle$ satisfying $J^2 |jm\rangle = j(j+1)|jm\rangle$

$$J_z |jm\rangle = m|jm\rangle.$$

we finally write down

$$|j_1 j_2 ; jm\rangle = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | jm\rangle |j_1 m_1\rangle |j_2 m_2\rangle$$

where $\langle j_1 m_1 j_2 m_2 | jm\rangle$ is call the C-G coefficient (Clebsch-Gordan)

Reversely, $|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{jm} |j_1 j_2 ; jm\rangle \langle jm | j_1 m_1 j_2 m_2\rangle$

$$\sum_{m_1 m_2} \langle jm | j_1 m_1 j_2 m_2 \rangle \langle j_1 m_1 j_2 m_2 | j' m' \rangle = \delta_{jj'} \delta_{mm'}$$

$$\sum_{jm} \langle j_1 m_1 j_2 m_2 | jm \rangle \langle jm | j_1 m_1' j_2 m_2' \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

§: The values of j.

$J_z |j_1 m_1 j_2 m_2\rangle = (m_1 + m_2) |j_1 m_1 j_2 m_2\rangle$. we have $(2j_1+1)(2j_2+1)$ bases.

① hence, the maxial vaule of $J_z = j_1 + j_2$, then the associated $J_{max} = j_1 + j_2$. This sector generates a set of $2(j_1+j_2)+1$ fold multiplets.

② Then for $J_z = j_1 + j_2 + 1$, there are 2 possible states.

$|j_1 j_1 - 1\rangle \otimes |j_2 j_2\rangle$, $|j_1 j_1\rangle \otimes |j_2 j_2 - 1\rangle$

The sector of $J = j_1 + j_2$ generates one $|j_1+j_2, j_1+j_2-1\rangle$

then there should exist another new state, whose J must be $j_1 + j_2 - 1$.

③ Then for the sector of $J_z = j_1 + j_2 - 2$, there exist 3 states $|j_1 j_1 - 2\rangle |j_2 j_2\rangle$ $|j_1 j_1 - 1\rangle |j_2 j_2 - 1\rangle$ $|j_1 j_1\rangle |j_2 j_2 - 2\rangle$

we have $J = j_1 + j_2$, $j_1 + j_2 - 1$, they yield 2 states with $J_z = j_1 + j_2 - 2$.

Hence, the one left must belong to the total $J = j_1 + j_2 - 2$.

we can repeat this procedure, until $J = |j_1 - j_2|$.

We count the # of states: (assume $j_1 > j_2$)

$J = j_1 + j_2 \rightarrow 2(j_1+j_2)+1$

$J = j_1 + j_2 - 1 \quad 2(j_1+j_2-1)+1$

$\vdots$

$J = j_1 - j_2 \quad 2(j_1-j_2)+1$

$\sum = (2j_1+2j_2+1 + 2j_1-2j_2+1) \times (j_1+j_2-j_1+j_2+1)/2$

$= (2j_2+1)(2j_1+1)$

$J=j_1+j_2$	$J=j_1+j_2-1,\ j_1+j_2-1$	
$\cancel{\lvert j_1 j_1\rangle \lvert j_2 j_2\rangle}$,	$\cancel{\lvert j_1 j_1-1\rangle \lvert j_2 j_2\rangle}$, ····	$\lvert j_1 -j_1\rangle \lvert j_2 j_2\rangle$
		$\vdots$
$\lvert j_1 j_1\rangle \cancel{\lvert j_2 j_2-1\rangle}$	$\lvert j_1 j_1-1\rangle \lvert j_2 j_2-1\rangle$	$\lvert j_1 -j_1\rangle \lvert j_2 j_2-1\rangle$
$\vdots$		$\vdots$
$\lvert j_1 j_1\rangle \lvert j_2 -j_2\rangle$	$\lvert j_0 j_1-1\rangle \lvert j_2 -j_2\rangle$ ···	$\lvert j_1 -j_1\rangle \lvert j_2 -j_2\rangle$

§ Example of spin 1/2, $1/2 \times 1/2 = 0 + 1$.

spin 1/2 $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{S} = \frac{\vec{\sigma}}{2}$. $\quad S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Consider two electrons each with spin 1/2, what's their total spin?

$$\vec{S} = \vec{S}_1 + \vec{S}_2 \quad \Rightarrow \quad S_- = S_{1-} + S_{2-}$$

① $|\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = |11\rangle$

$$(S_{1-} + S_{2-})\ |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 = S_- |11\rangle = \sqrt{(1+1)(1-1+1)}\ |10\rangle$$

② $|\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 = \sqrt{2}\,|10\rangle$

$$(S_{1-} + S_{2-})\ (|\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2) = \sqrt{2}\, S_- |10\rangle$$

$$|\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 = \sqrt{2}\,\sqrt{(1-0)(1-0+1)}\,|1-1\rangle$$

$$\Rightarrow \quad |\tfrac{1}{2}-\tfrac{1}{2}\rangle_1 |\tfrac{1}{2}-\tfrac{1}{2}\rangle_2 = |1-1\rangle$$

Summary: ← in simplified notation

$$\begin{cases} |11\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2 \\ |10\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2) \\ |1-1\rangle = |\downarrow\rangle_1 |\downarrow\rangle_2 \end{cases}$$

Then we have another state with $S_z = 0$, which is orthogonal to $|10\rangle$. This state should belong to $|00\rangle$. $\propto |\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2$

But we need to assign a sign convention, we require that

$$\langle j' j_z^{(12)} | j_{1z} | j j_z^{(12)} \rangle \geq 0.$$

Here, $|j j_z^{(12)}\rangle$ means the construction by a fixed sequence of J_1 and J_2. If we switch the sequence to be J_2 and J_1, then the consequence is denoted as $|j j_z^{(21)}\rangle$.

According to this convention, we have

$$\boxed{|00\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)}$$

such that $\langle 00|\, j_{1z}\, |10\rangle = \frac{1}{2}\left(\langle\uparrow|_1\langle\downarrow|_2 - \langle\downarrow|_1\langle\uparrow|_2\right)$

$$\cdot\left(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2\right) > 0$$

Then we have obtained all the $\langle 1m|\,\frac{1}{2}s_z\,\frac{1}{2}s_z'\rangle$.

CG coefficients of

§ how about $1 \times 1/2 = 3/2 \oplus 1/2$

$$|3/2\ 3/2\rangle = |1\,1\rangle_1 \otimes |1/2\ 1/2\rangle_2$$

$\downarrow J_-$

$$|3/2\ 1/2\rangle = \sqrt{2/3}\,|1\,0\rangle_1|1/2\ 1/2\rangle_2 + \sqrt{1/3}\,|1\,1\rangle_1|1/2\ -1/2\rangle_2$$

$\downarrow J_-$

$$|3/2\ -1/2\rangle = \sqrt{1/3}\,|1\,-1\rangle_1|1/2\ 1/2\rangle_2 + \sqrt{2/3}\,|1\,0\rangle_1|1/2\ -1/2\rangle_2$$

$\downarrow J_-$

$$|3/2\ -3/2\rangle = |1\,-1\rangle_1 \otimes |1/2\ -1/2\rangle_2$$

another state with $J_z = 1/2$, $\sqrt{1/3}\,|1\,0\rangle_1|1/2\ 1/2\rangle_2 - \sqrt{2/3}\,|1\,1\rangle_1|1/2\ -1/2\rangle$

$$j_{1z}\,|3/2\ 1/2\rangle = \sqrt{1/3}\,|1\,1\rangle_1|1/2\ -1/2\rangle_2$$

$$\langle 1/2\ 1/2|\, j_{2z}\,|3/2\ 1/2\rangle > 0 \Rightarrow |1/2\ 1/2\rangle = \sqrt{2/3}\,|1\,1\rangle_1|1/2\ -1/2\rangle_2 - \sqrt{1/3}\,|1\,0\rangle_1|1/2\ 1/$$

$$|1/2\ -1/2\rangle = \sqrt{1/3}\,|1\,0\rangle_1|1/2\ -1/2\rangle_2 - \sqrt{2/3}\,|1\,-1\rangle_1|1/$$

Conside the angular momentum addition of spherical harmonics $Y_{lm}(\Omega)$ and electron spin. $\vec{j} = \vec{L} + \vec{S}$

Hence $j_\pm = l \pm 1/2$.

For the sector of $j_+ = l + 1/2$, we ~~start~~ with the ~~state~~ $|j_+ j_z = j_+\rangle$

$$|j_+, j_+\rangle = Y_{lm}(\Omega) \otimes |\uparrow\rangle = Y_{lm}(\Omega) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} Y_{lm}(\Omega) \\ 0 \end{pmatrix}$$

For the general case $|j_+ j_z\rangle = a_m Y_{lm}(\Omega) \otimes |\uparrow\rangle + b_m Y_{l,m+1} \otimes |\downarrow\rangle$

$j_z = m + 1/2$

$$= \begin{pmatrix} a_m Y_{lm} \\ b_m Y_{l,m+1} \end{pmatrix}$$

$J_\pm = L_\pm + S_\pm$

$$\Rightarrow J_- |j_+ j_z\rangle = \sqrt{(j_+ + j_z)(j_+ - j_z + 1)}\ |j_+ j_z - 1\rangle =$$

$$J_- |j_+ j_z\rangle = a_m \sqrt{(l+m)(l-m+1)}\ Y_{l,m-1}(\Omega) \otimes |\uparrow\rangle$$

$$+ a_m Y_{lm} \otimes |\downarrow\rangle$$

$$+ b_m \sqrt{(l+m+1)(l-m-1+1)}\ Y_{l,m}(\Omega) \otimes |\downarrow\rangle$$

$$J_- |j_+ j_z\rangle = \sqrt{(l+1/2+m+1/2)(l-m+1)}\ \left(a_{m-1} Y_{l,m-1} \otimes |\uparrow\rangle + b_{m-1} Y_{lm} \otimes |\downarrow\rangle\right)$$

$$\Rightarrow \begin{cases} a_m \sqrt{(l+m)(l-m+1)} = a_{m-1} \sqrt{(l+m+1)(l-m+1)} \\ a_m + b_m \sqrt{(l+m+1)(l-m)} = \sqrt{(l+m+1)(l-m+1)}\ b_{m-1} \end{cases}$$

$$\Rightarrow \quad \frac{a_{m-1}}{a_m} = \sqrt{\frac{l+m}{l+m+1}} \qquad m = l, l-1, \cdots -l$$

$$\begin{cases} a_l = 1 \end{cases}$$

$$\Rightarrow \quad a_m = \sqrt{\frac{l+m+1}{2l+1}} \qquad \Rightarrow \quad \sqrt{\frac{l+m+1}{2l+1}} + b_m \sqrt{(l+m+1)(l-m)}$$

$$= b_{m-1} \sqrt{(l+m+1)(l-m+1)}$$

$$\Rightarrow \begin{cases} \sqrt{\frac{1}{2l+1}} + b_m \sqrt{l-m} = b_{m-1} \sqrt{(l-m)+1} \\ b_l = 0 \end{cases}$$

$$\Rightarrow \quad b_{l-1} = \sqrt{\frac{1}{2l+1}} \quad \cdots \quad \Rightarrow \quad b_m = \sqrt{\frac{l-m}{2l+1}}$$

Check $\Rightarrow \sqrt{\frac{1}{2l+1}} \left[1 + l - m \right] = \sqrt{\frac{1}{2l}} \left[l - m + 1 \right)$

organize $\Rightarrow$

$$\boxed{\left| j_+, j_z = m + 1/2 \right\rangle = \begin{pmatrix} \sqrt{\frac{l+m+1}{2l+1}}\, Y_{lm} \\ \sqrt{\frac{l-m}{2l+1}}\, Y_{l,m+1} \end{pmatrix}}$$

For another sector $j_- = l - 1/2$

$$\left| j_-, j_z = m + 1/2 \right\rangle = \begin{pmatrix} c_m\, Y_{lm} \\ d_m\, Y_{l,m+1} \end{pmatrix}$$

$$\left\langle j_+ \, j_z \,\middle|\, j_- \, j_z \right\rangle = 0 \Rightarrow \sqrt{\frac{l+m+1}{2l+1}}\, c_m + \sqrt{\frac{l-m}{2l+1}}\, d_m = 0$$

$$C_m/d_m = -\sqrt{\frac{l-m}{l+m+1}}$$

$$\Rightarrow \quad C_m = -\sqrt{\frac{l-m}{2l+1}}\, e^{i\varphi}$$

$$d_m = \sqrt{\frac{l+m+1}{2l+1}}\, e^{i\varphi}$$

$\gg$ up to a phase

$$\Rightarrow \boxed{|j_-, j_z = m+1/2\rangle = \begin{pmatrix} -\sqrt{\frac{l-m}{2l+1}}\, Y_{lm} \\ \sqrt{\frac{l+m+1}{2l+1}}\, Y_{l,m+1} \end{pmatrix} e^{i\varphi}}$$

phase requirement

$$\langle j_+ \; j_z = m+1/2 \,|\, l_z \,|\, j_- \; j_z = m+1/2 \rangle = \frac{\sqrt{(l-m)(l+m+1)}}{2l+1} \left[-m + m + 1 \right] e^{i\varphi} > 0$$

$$\Rightarrow \varphi = 0$$