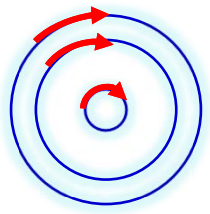
Lecture 16: Application of angular momentum --3D Landau level

Landau levels (LL): complex analyticity

Eigenstates in the B-field: $H\psi = E_n\psi$



$$H_{2DLL} = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 \quad \vec{A} = \frac{\vec{B}}{2}\times\vec{r}$$

Lowest Landau level (LLL) wavefunctions:

$$\psi_{LLL}(x,y) = f(z,\bar{z})e^{-\frac{|z|^2}{4l^2}}$$

$$D_z\psi_{LLL}(z,\bar{z}) = 0 \qquad D_z = \partial_z - \frac{\bar{z}}{4l_B^2}$$

$$\partial_{\bar{z}}f(z,\bar{z}) = 0 \Rightarrow f(z,\bar{z}) \to f(z) = 1, z, z^2, \dots, z^n, \dots$$

1

1

Harmonic oscillators

$$H_{2DLL} = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 \qquad \vec{A} = \frac{\vec{B}}{2}\times\vec{r}$$

The orbital Zeeman term : $L_z = (\vec{r}\times\vec{p})\cdot\hat{z}$

$$H_{2D\,LL} = \frac{p^2}{2M} + \frac{1}{2}M\omega^2r^2 \mp \omega L_z, \qquad \omega = \frac{e|B|}{2Mc}, l_B = \sqrt{\frac{hc}{eB}}$$

- ωL_z commutes with 2D harmonic oscillator Hamiltonian
- Reorganize the harmonic oscillator wavefunctions → LL levels

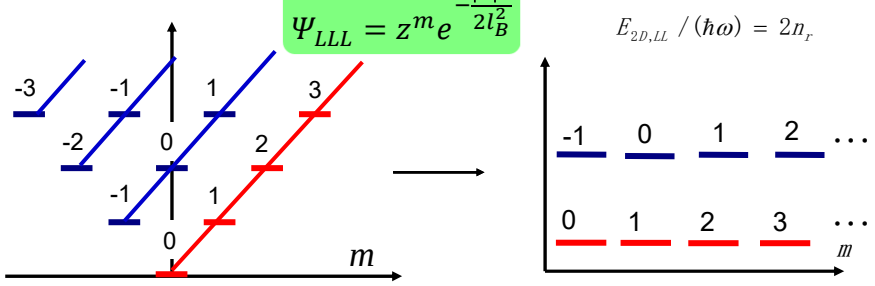
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Topology: only $m \geq 0$ are selected

$E_{2D,HO}/(\hbar\omega) = 2n_r + |m| + 1$

$\Psi_{LLL} = z^m e^{-\frac{|z|^2}{2l_B^2}}$



- If viewed horizontally, they are topologically trivial.
- If viewed along the diagonal line, they become LLs.

$$E_{Zeeman}/(\hbar\omega) = -m$$

3

3

Criterion of selection: complex analyticity

- Landau levels are elegant and explicit.
  1. The 2D ordinary QM WF $\psi(x, y)$ belongs to real analysis
  2. Cauchy-Riemann condition → complex analyticity (chirality).
  3. Chirality is physically imposed by the B-field.
- Laughlin wavefunction – Nobel Prize of Physics, 1998.

$$\Psi_{sym}(z_1, z_2, \cdots, z_n) = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i \frac{|z_i|^2}{4l_B^2}}$$

Analytic function of multiple complex variables.

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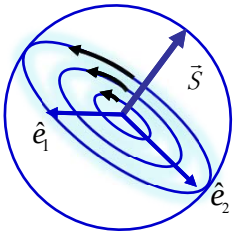
High dimensions: challenge and our stragetgy

1. 3D harmonic wavefunctions.
2. Selection criterion: quaternionic analyticity
3. Physically imposed by spin-orbit coupling.

Symmetric-like gauge (3D quantum top):

Frame $(\hat{e_1}, \hat{e}_2, \hat{e}_3)$: the complex plane is floating and subject to rotation.

Generalizable to higher dimensions.



5

3D – Aharanov-Casher potential

- The SU(2) gauge potential:

2D: $\vec{A} = \frac{1}{2} B\hat{z} \times \vec{r}$ → 3D: $\vec{A}_{\alpha\beta} = \frac{g}{2} \vec{\sigma}_{\alpha\beta} \times \vec{r}$

- 3D LL Hamiltonian = harmonic oscillator + spin-orbit coupling.

$$H_{LL}^{3D} = \frac{1}{2M}\left(\vec{P} - \frac{e}{c}\vec{A}_{\alpha\beta}\right)^2 - \frac{M}{2}\omega^2 r^2$$

$$= \frac{P^2}{2M} + \frac{1}{2}M\omega^2 r^2 - \omega\vec{\sigma}\cdot\vec{L} \qquad \omega = \frac{|eg|}{2Mc} \rightarrow l_g = \sqrt{\frac{hc}{|eg|}}$$

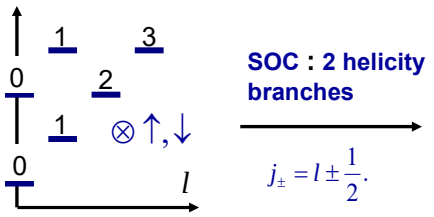
- The full 3D rotational symm. + **time-reversal symm**.

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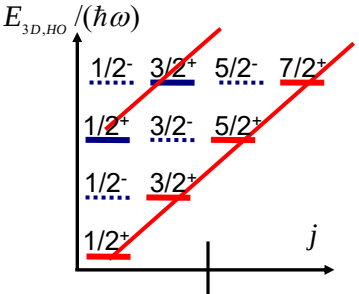
Constructing 3D Landau Levels

$E_{3D,HO}/(\hbar\omega) = 2n_r + l + \frac{3}{2}$



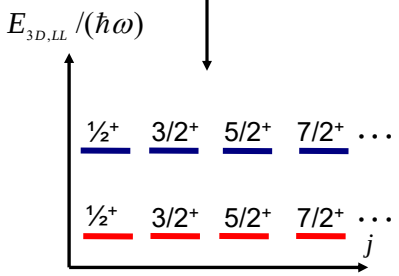
SOC : 2 helicity branches

$j_\pm = l \pm \frac{1}{2}.$



$$H_{LL}^{3D} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 - \omega\vec{\sigma}\cdot\vec{L}$$

$$\vec{\sigma}\cdot\vec{L} = \begin{cases} l\hbar & \text{for } j_+ \\ -(l+1)\hbar & \text{for } j_- \end{cases}$$

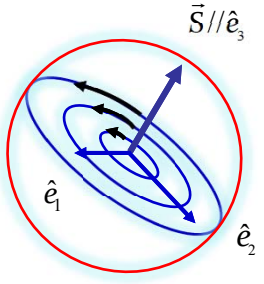


7

The coherent state picture for 3D LLL WFs

- The highest weight state $j_z = j_+$

$$\psi^{LLL}_{j_+, j_z = j_+}(\vec{r}) = \begin{pmatrix} (x+iy)^l \\ 0 \end{pmatrix} e^{-\frac{r^2}{4l^2}}$$



- Complex analyticity in terms of $\hat{e}_1 + i\vec{e}_2$

$$\psi^{LLL}(\vec{r}) = [(\hat{e}_1 + i\vec{e}_2)\cdot\vec{r}]^l \otimes \chi_{\hat{e}_3}$$

- Helicity: Rotation 180^o → $[(\hat{e}_1 - i\vec{e}_2)\cdot\vec{r}]^l \otimes \chi_{-\hat{e}_3}$

- N-dimensions: Pick up any two axes and define a complex plane with a spin-orbit coupled helical structure.

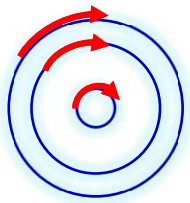
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8

From real to complex to quarternionic

- 1D harmonic levels: real polynomials.
- 2D LLs: complex analytic polynomials.

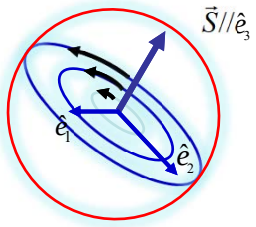
$$\psi_{2D}^{LLL}(\vec{r}) = z^m, z = x + iy, m \geq 0$$



U(1) Phase

- 3D LLs: SU(2) group space → **quaternionic** analytic polynomials.

$$\psi_{3D}^{LLL}(\vec{r}) = [(\hat{e}_1 + i\vec{e}_2) \cdot \vec{r}]^l \otimes \chi_{\hat{e}_3}$$



c.f. a top, SU(2) 9

9

Analyticity condition ←→ Weyl equation (Euclidean)

2D complex analyticity ⇨ 1D chiral edge mode

$$\psi(x,y) = f(z)e^{-\frac{|z|^2}{4l^2}} \qquad \psi(t,x) = f(x-t)$$

$$(\partial_x + i\partial_y)f = 0 \qquad (\partial_t - \partial_x)\psi = 0$$

4D: quaternionic analyticity ⇨ 3D Weyl boundary mode

$$(\partial_x + i\partial_y + j\partial_z + k\partial_u)f = 0 \qquad (\partial_t + \sigma_x\partial_x + \sigma_y\partial_y + \sigma_z\partial_z)\psi = 0$$

3D: quaternionic analyticity ⇨ 2D helical Dirac surface mode

$$(\partial_x + i\partial_y + j\partial_z)f = 0 \qquad (\partial_t + \sigma_x\partial_x + \sigma_y\partial_y)\psi = 0$$

10

10

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Landau level, & Zero modes

Consider the Landau level problem in a uniform magnetic field $\vec{B}$.

In the symmetric gauge $\vec{A} = \frac{B}{2}\hat{z}\times\vec{r} = \frac{B}{2}(-y, x)$.

$$H = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} = \frac{p^2}{2m} + \frac{m}{2}\left(\frac{eB}{2mc}\right)^2 r^2 - \frac{eB}{2mc}\hat{z}\cdot(\vec{r}\times\vec{p})$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 r^2 \mp \omega_0 L_z \quad , \text{ where } \omega_0 = \left|\frac{eB}{2mc}\right|.$$

Define $l = \sqrt{\frac{\hbar}{m\omega_0}} = \sqrt{\frac{2\hbar c}{eB}} = \sqrt{2}\, l_B$, where $l_B = \sqrt{\frac{\hbar c}{eB}}$.

$$a_x = \frac{1}{\sqrt{2}}\left(x/l + \frac{ip_x\cdot l}{\hbar}\right), \qquad a_y = \frac{1}{\sqrt{2}}\left(y/l + \frac{ip_y l}{\hbar}\right)$$

$$a_x^\dagger = \frac{1}{\sqrt{2}}\left(x/l - \frac{ip_x\cdot l}{\hbar}\right), \qquad a_y^\dagger = \frac{1}{\sqrt{2}}\left(y/l - \frac{ip_y l}{\hbar}\right)$$

$$L_z = xp_y - yp_x = -i\hbar\left[a_x^\dagger a_y - a_y^\dagger a_x\right]$$

$$\Rightarrow H = \hbar\omega_0\left[a_x^\dagger a_x + a_y^\dagger a_y + 1\right] + i\hbar\omega_0\left[a_x^\dagger a_y - a_y^\dagger a_x\right]$$

$$= 2\hbar\omega_0\left[\left(\frac{a_x^\dagger \mp i a_y^\dagger}{\sqrt{2}}\right)\left(\frac{a_x \pm i a_y}{\sqrt{2}}\right) + 1/2\right]$$

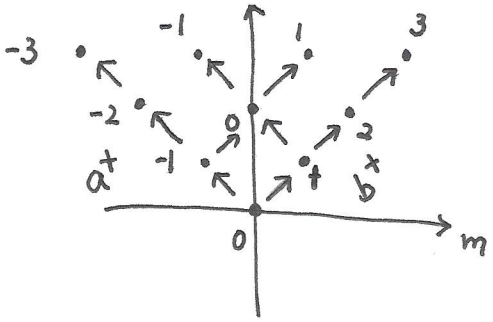
Define $a = \dfrac{a_x + i a_y}{\sqrt{2}}$, $\omega_c = \dfrac{eB}{mc} \Rightarrow H = \hbar\omega_c\left[a^\dagger a + \frac{1}{2}\right]$

$$b = \frac{a_x \mp i a_y}{\sqrt{2}}$$

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$$\begin{cases} [L_z,\ b^\dagger] = -\frac{i\hbar}{\sqrt{2}}[a_x^\dagger a_y - a_y^\dagger a_x,\ a_x^\dagger + i a_y^\dagger] = -\frac{i\hbar}{\sqrt{2}}\left[(+i)\,a_x^\dagger - a_y^\dagger\right] = +b^\dagger \\ [L_z,\ a^\dagger] = -a^\dagger \end{cases}$$

2D Harmonic oscillator energy levels





$b^\dagger$: increase angular momentum m but within the same Landau level!

$a^\dagger$: decrease angular momentum, increase Landau level index.

☆ LLL (lowest Landau level as the zero modes)

$$a_x + i a_y = \frac{1}{\sqrt{2}}\left(\frac{x+iy}{\ell} + \frac{i(p_x + i p_y)\cdot\ell}{\hbar}\right) = \frac{1}{\sqrt{2}}\left(\frac{x+iy}{\ell} + (\partial_x + i\partial_y)\cdot\ell\right)$$

define $z = x + iy$

$\bar{z} = x - iy$

$\to x = \frac{1}{2}(z+\bar{z})$

$y = \frac{-i}{2}(z-\bar{z})$

$$\partial_z = \frac{\partial x}{\partial z}\partial_x + \frac{\partial y}{\partial z}\partial_y = \frac{1}{2}(\partial_x - i\partial_y)$$

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

$$\Rightarrow\quad a_x + i a_y = \frac{1}{\sqrt{2}}\left(\frac{z}{\ell} + 2\ell\,\partial_{\bar{z}}\right)$$

$$(a_x + i a_y)\,\psi(z,\bar{z}) = 0 \quad\Rightarrow\quad \left(\partial_{\bar{z}} + \frac{z}{2\ell^2}\right)\psi(z,\bar{z}) = 0$$

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Plug in $\psi(z,\bar{z}) = f(z,\bar{z})\, e^{-\frac{z\bar{z}}{2l^2}}$ ←

$$\partial_{\bar{z}}\left[f(z,\bar{z})\, e^{-z\bar{z}/2l^2}\right] = \left[\partial_{\bar{z}} f(z,\bar{z}) + f(z,\bar{z})\,\frac{-z}{2l^2}\right] e^{-|z|^2/2l^2}$$

$$\Rightarrow \quad \partial_{\bar{z}} f(z,\bar{z}) = 0, \quad \text{i.e.} \quad \boxed{f(z,\bar{z}) = f(z),}$$

✲ The lowest Landau level WF is a complex analytic function!

$$\psi_m = \frac{1}{\sqrt{2\pi l_B^2\, 2^m\, m!}} \left(\frac{z}{l_B}\right)^m e^{-\frac{1}{4}\frac{|z|^2}{l_B^2}}$$

✲ How about the singular function?

$$\partial_{\bar{z}}\left(\frac{1}{z}\right) = \partial_z\left(\frac{1}{\bar{z}}\right) = \pi\,\delta^{(2)}(x,y)$$

we have $\int_M d^2x\, \partial_{\bar{z}} f(z) = -\frac{i}{2}\oint_{\partial M} dz\, f(z)$ ← explicitly calculation verification!

then $\frac{1}{\pi}\int_M d^2x\, \partial_{\bar{z}}\left(\frac{f(z)}{z}\right) = \frac{1}{2\pi i}\oint_{\partial m} dz\, \frac{f(z)}{z} = f(0)$ ← Cauchy

on the other hand $\frac{1}{\pi}\int_M d^2x\, \partial_{\bar{z}}\left(\frac{f(z)}{z}\right) = \frac{1}{\pi}\int d^2x\, f(z)\, \partial_{\bar{z}}\left(\frac{1}{z}\right)$

Hence $\boxed{\partial_{\bar{z}}\left(\frac{1}{z}\right) = \pi\,\delta^{(2)}(x,y), \text{ similarly}}$

$$\boxed{\partial_z\left(\frac{1}{\bar{z}}\right) = \pi\,\delta^{(2)}(x,y)}$$

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Similarly $\partial_{\bar{z}}\left(\frac{1}{z^n}\right) = ?$

$$\frac{1}{\pi}\int_M d^2x\ \partial_{\bar{z}}\left(\frac{f(z)}{z^n}\right) = \frac{1}{2\pi i}\oint_{\partial M} dz\ \frac{f(z)}{z^n} = \frac{1}{2\pi i}\oint_{\partial M} dz\ \frac{f^{(n-1)}(z)}{(n-1)!}\ \frac{z^{n-1}}{z^n}$$

$$= \frac{1}{(n-1)!} f^{(n-1)}(z)\Big|_{z=0}$$

Then $\frac{1}{\pi}\int_M d^2x\ \partial_{\bar{z}}\left(\frac{f(z)}{z^n}\right) = \frac{1}{\pi}\int d^2x\ f(z)\ \partial_{\bar{z}}\left(\frac{1}{z^n}\right)$

$$\Rightarrow \partial_{\bar{z}}\left(\frac{1}{z^n}\right) = \pi\ \frac{\delta^{(2)}(x,y)}{(n-1)!}\cdot\frac{d^{n-1}}{dz^{n-1}}$$

Similarly $\partial_z\left(\frac{1}{\bar{z}^n}\right) = \pi\,\delta^{(2)}(x,y)\ \frac{1}{(n-1)!}\ \frac{d^{n-1}}{d\bar{z}^{n-1}}$

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zero modes of Dirac electron in a flux Φ.

* Consider $H = c\sigma_x(p_x - \frac{e}{c}A_x) + c\sigma_y(p_y - \frac{e}{c}A_y)$ ① uniform B-field.

$$= c\begin{bmatrix} 0 & p_x - \frac{e}{c}A_x - i(p_y - \frac{e}{c}A_y) \\ p_x - \frac{e}{c}A_x + i(p_y - \frac{e}{c}A_y), & 0 \end{bmatrix}$$

$$A_x = -\frac{By}{2}$$
$$A_y = \frac{B}{2}x$$

$$H = \frac{\hbar c}{\ell}\begin{bmatrix} 0, & i(-i\ell\frac{p_x}{\hbar} + \frac{e}{c}A_y\frac{\ell}{\hbar}) - i\ell\frac{p_y}{\hbar} - \frac{e}{c}A_x\frac{\ell}{\hbar} \\ & -i(i\ell\frac{p_x}{\hbar} + \frac{e}{c}A_y\frac{\ell}{\hbar}) + i\ell\frac{p_y}{\hbar} - \frac{e}{c}A_x\frac{\ell}{\hbar}, \quad 0 \end{bmatrix}$$

$$\frac{e}{c}A_y\frac{\ell}{\hbar} - i\ell\frac{p_x}{\hbar} = \frac{eB}{2\hbar c}\ell\cdot x - i\ell\frac{p_x}{\hbar}, \quad \text{set } \ell = \sqrt{\frac{2\hbar c}{eB}}$$

$$= \frac{x}{\ell} - i\ell\frac{p_x}{\hbar} = \sqrt{2}\,a_x^\dagger$$

$$-\frac{e}{c}A_x\frac{\ell}{\hbar} - i\ell\frac{p_y}{\hbar} = \frac{eB}{2\hbar c}\ell\cdot y - i\ell\frac{p_y}{\hbar} = \frac{y}{\ell} - i\ell\frac{p_y}{\hbar} = \sqrt{2}\,a_y^\dagger$$

$$\Rightarrow H = \frac{\sqrt{2}\hbar c}{\ell}\begin{bmatrix} 0 & i\,a_x^\dagger + a_y^\dagger \\ -i\,a_x + a_y, & 0 \end{bmatrix} = \frac{2\hbar c}{\ell}\begin{bmatrix} 0 & \frac{i(a_x^\dagger - i a_y^\dagger)}{\sqrt{2}} \\ -\frac{i(a_x + i a_y)}{\sqrt{2}} & 0 \end{bmatrix}$$

Hence, the zero mode

$$\begin{bmatrix} 0 & i\,a^\dagger \\ -i\,a & 0 \end{bmatrix}\begin{bmatrix} \psi \\ 0 \end{bmatrix} = 0 \Rightarrow (a_x + i a_y)\psi = 0$$

which is the same as the non-relavistic case.

$$a = \frac{1}{\sqrt{2}}(a_x + i a_y)$$

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② In general, we don't need B to be a constant to arrive at zero mode

$$\frac{i l P_x}{\hbar} + \frac{el}{\hbar c} A_y = l\left[\partial_x + \frac{e}{\hbar c} A_y\right] = l\left[\partial_x + \frac{2\pi}{\Phi_0} A_y\right] = l\left[\partial_x + g_x\right]$$

$$\frac{i l P_y}{\hbar} - \frac{e}{c} A_x \frac{l}{\hbar} = l\left[\partial_y - \frac{2\pi}{\Phi_0} A_x\right] = l\left[\partial_y + g_y\right]$$

we chose the Coulomb gauge that $\partial_x A_x + \partial_y A_y = 0$.

$\Rightarrow \quad -\partial_x g_y + \partial_y g_x = 0 \Rightarrow (\nabla \times \vec{g})_z = 0$, where $\vec{g} = \frac{2\pi}{\Phi_0}(A_y, -A_x)$

Hence $\vec{g} = \nabla \text{Ⓗ}$, i.e $g_x = \partial_x \text{Ⓗ}$, $g_y = \partial_y \text{Ⓗ}$

$\Rightarrow$ then zero mode satisfies

$$(\partial_x + i\partial_y + g_x + i g_y)\psi = 0$$

set $\psi = f(z,\bar{z})\, e^{-\text{Ⓗ}(x,y)}$

$$(\partial_x + i\partial_y)\psi = \left[2\partial_{\bar{z}} f(z,\bar{z}) - \partial_x \text{Ⓗ} - i\partial_y \text{Ⓗ} + g_x + i g_y\right] e^{-\text{Ⓗ}} = 0$$

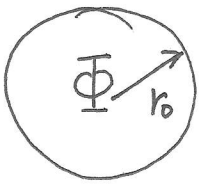
$$\Rightarrow \quad \partial_{\bar{z}} f(z,\bar{z}) = 0, \qquad \Rightarrow f(z,\bar{z}) = f(z).$$

Hence, up to the envelop function $\text{Ⓗ}(x,y)$, satisfying $\nabla \text{Ⓗ} = \frac{2\pi}{\Phi_0} \vec{A} \times \hat{z}$.

the zero mode wavefunctions are the same as in the uniform B-field case!

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Consider, there's a total flux Φ inside $r < r_0$.



As $r \to \infty$, $\oint \vec{A} \cdot d\vec{r} = \oint \vec{g} \cdot d\hat{n} = \Phi$

$$\vec{A} = \frac{\Phi}{2\pi r}\hat{e}_\theta \quad \to \quad \vec{g} = \frac{2\pi}{\Phi_0}\vec{A} \times \hat{z} = \frac{\Phi}{\Phi_0 r}\hat{e}_r$$

$$\nabla \textcircled{H} = \vec{g} \quad \Rightarrow \quad \frac{d}{dr}\textcircled{H}(r) = \frac{\Phi}{\Phi_0}\frac{1}{r}$$

$$\Rightarrow \quad \textcircled{H}(r) = \frac{\Phi}{\Phi_0}\ln\frac{r}{r_0} = \ln\left(\frac{r}{r_0}\right)^{\Phi/\Phi_0}$$

$$\Rightarrow \quad \psi = f(z)\left(\frac{r_0}{r}\right)^{\Phi/\Phi_0} \quad \text{as } |z| = r \to \infty$$

in order for $|\psi|^2$ to be integratable. $f(z) = z^n$

$$\Rightarrow \int r\,dr\,|\psi| = \int_0^{+\infty} r \cdot r^{2(n-\Phi/\Phi_0)}\,dr \quad \to \text{converge}$$

$$\Rightarrow \quad r^{1+2(n-\Phi/\Phi_0)}$$

decay faster than r^{-1}

or $\quad 2(n-\Phi/\Phi_0) < -2$

or $\quad n - \Phi/\Phi_0 < -1$.

$$\Rightarrow \boxed{n = 0, 1, 2, \ldots \quad [\Phi/\Phi_0] - 1}$$

# of zero modes.

(*) for non-zero mode $H\psi = E\psi$, then $H(\sigma_z\psi) = -\sigma_z H\psi = -(E\psi)$.

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Zero energy modes in high dimensions

Define $b = \sigma_i a_i$, where $a_i = \frac{1}{\sqrt{2}}\left(\frac{x_i}{\ell} + i\frac{P_i \cdot \ell}{\hbar}\right)$. Then

$$H = \hbar\omega\, b^+ b = \hbar\omega\,[\sigma_i a_i^+][\sigma_j a_j]$$

$$\Rightarrow H/\hbar\omega = a_i^+ a_i + \sum_{i<j} \sigma_i\sigma_j\left(a_i^+ a_j - a_j^+ a_i\right) \qquad \sigma_i\sigma_j = i\sigma_k$$

$$-i\left(a_i^+ a_j - a_j^+ a_i\right) = L_k$$

$$= a_i^+ a_i - \sigma_k L_k .$$

Zero-mode

$$\sigma_i\left(\frac{iP_i\ell}{\hbar} + \frac{x_i}{\ell}\right)\psi(\vec{r}) = 0$$

$$\sigma_i\left(\partial_i + \ell^{-2}x_i\right)\psi(\vec{r}) = 0 .$$

plug in $\psi(\vec{r}) = f(\vec{r})\, e^{-r^2/2\ell^2}$

$$\sigma_i\left(\partial_i f + \ell^{-2}x_i f + (-)\frac{x_i}{\ell^2} f\right) e^{-r^2/2\ell^2} = 0$$

$$\Rightarrow \boxed{\sigma_i \cdot \partial_i f = 0}$$ — we seek polynominal solutions

① Quaternion Rep

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rightarrow f_1 + jf_2 = \mathrm{Re}f_1 + i\,\mathrm{Im}f_1 + j\,\mathrm{Re}f_2 - k\,\mathrm{Im}f_2$$

Compare complex Rep $\begin{pmatrix} f_1 & -f_2^* \\ f_2 & f_1^* \end{pmatrix} = \mathrm{Re}f_1 + i\sigma_z\,\mathrm{Im}f_1 - i\sigma_y\,\mathrm{Re}f_2 + i\sigma_x\,\mathrm{Im}f_2$

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quaternions :

$$\begin{aligned} i &\longrightarrow i\sigma_z \\ j &\longrightarrow -i\sigma_y \\ k &\longrightarrow -i\sigma_x \end{aligned}$$

$ij = k \longrightarrow i\sigma_z(-i)\sigma_y = -i\sigma_x$

the product rule is maintained!

Then

$$\sigma_x\partial_x + \sigma_y\partial_y + \sigma_z\partial_z = \sigma_x[\partial_x + i\sigma_z\partial_y - i\sigma_y\partial_z]$$

$$\partial_x + i\sigma_z\partial_y - i\sigma_y\partial_z = \partial_x + i\partial_y + j\partial_z .$$

$$\sigma_i\cdot\partial_i\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \Rightarrow \sigma_i^*\,\partial_i\begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix} = 0$$

$$i\sigma_2\,\sigma_i^*(-i)\sigma_2\,\partial_i(i\sigma_2)\begin{pmatrix} f_1^* \\ f_2^* \end{pmatrix} = 0 \Rightarrow -\sigma_i\cdot\partial_i\begin{pmatrix} -f_2^* \\ f_1^* \end{pmatrix} = 0$$

$$\Rightarrow \sigma_i\cdot\partial_i\begin{pmatrix} f_1 & -f_2^* \\ f_2 & f_1^* \end{pmatrix} = 0$$

$$\Rightarrow \sigma_x[\partial_x + i\sigma_z\partial_y - i\sigma_y\partial_z][\mathrm{Re}f_1 + i\sigma_z\mathrm{Im}f_1 - i\sigma_y\mathrm{Re}f_2 + i\sigma_x\mathrm{Im}f_2]$$

$$\rightarrow [\partial_x + i\partial_y + j\partial_z][\underbrace{\mathrm{Re}f_1 + i\mathrm{Im}f_1 + j\mathrm{Re}f_2 - k\mathrm{Im}f_2}_{f_1 + jf_2}] = 0$$

$$\Rightarrow \boxed{\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z}\right]f = 0}$$

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Hw about 4D ?

$$b = a_4 + i\sigma_i a_i \quad , \quad b^\dagger = a_4^\dagger - i\sigma_i a_i^\dagger$$

$$H = \hbar\omega\, b^\dagger b \Rightarrow H/\hbar\omega = (a_4^\dagger - i\sigma_i a_i^\dagger)(a_4 + i\sigma_j a_j)$$

$$= a_4^\dagger a_4 + (\sigma_i a_i^\dagger)(\sigma_j a_j) + i\sigma_j\left[a_4^\dagger a_j - a_j^\dagger a_4\right]$$

$$H/\hbar\omega = \sum_{i=1}^{4} a_i^\dagger a_i + i\sum_{k=1}^{3}\sigma_k \underbrace{a_i^\dagger a_j}_{\epsilon_{kij}} + i\sum_{j=1}^{3}\sigma_j\left(a_4^\dagger a_j - a_j^\dagger a_4\right)$$

$$= \sum_{i=1}^{4} a_i^\dagger a_i - \left(l_{12}\sigma_3 + l_{23}\sigma_1 + l_{31}\sigma_2 + l_{41}\sigma_1 + l_{42}\sigma_2 + l_{43}\sigma_3\right]$$

$$(a_4 + i\sigma_i a_i)\,\psi(r) = 0$$

$$\left(\frac{iP_4 l}{\hbar} + \frac{x_4}{l}\right) + i\sigma_i\left(\frac{iP_i l}{\hbar} + \frac{x_i}{l}\right)$$

$$\left[\partial_4 + i\sigma_i\partial_i + \left[x_4 + i\sigma_i x_i\right] l^{-2}\right]\psi(r) = 0$$

$$\psi(r) = f(r)\, e^{-r^2/2l^2} \quad , \quad (\partial_4 + i\sigma_i\partial_i)\, e^{-r^2/2l^2}$$

$$= -\frac{1}{l^2}\left[x_4 + i\sigma_i x_i\right] e^{-r^2/2l^2}$$

$$\Rightarrow \boxed{\left[\partial_4 + i\sigma_i\partial_i\right] f(r) = 0}$$

$$\left[\partial_x + i\sigma_z\partial_y - i\sigma_y\partial_z - i\sigma_x\partial_4\right]\begin{pmatrix} f_1 & -f_2^* \\ f_2 & f_1^* \end{pmatrix} = 0$$

$$\Rightarrow \boxed{(\partial_x + i\partial_y + j\partial_z + k\partial_4)\, f = 0, \quad \text{where } f = f_1 + jf_2.}$$

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Connection to spin-orbit coupled harmonics

$$\sigma_i \partial_i = \begin{pmatrix} \partial_z & \partial_x - i\partial_y \\ \partial_x + i\partial_y & -\partial_z \end{pmatrix} \Rightarrow \sigma_i \partial_i f = 0, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} (\partial_x + i\partial_y) f_1 = \partial_z f_2 \\ (\partial_x - i\partial_y) f_2 = -\partial_z f_1 \end{cases} \longrightarrow \nabla^2 f_{1,2} = 0.$$

If we suppress f_2, we have $f_1 = (x+iy)^l$, which reduces back to complex analytic function.

$$\vec{\sigma}\cdot\vec{p} = \frac{(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{r})}{r^2}(\vec{\sigma}\cdot\vec{p}) = \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[\vec{r}\cdot\vec{p} + i\vec{\sigma}\cdot(\vec{r}\times\vec{p})\right]$$

$$= \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[-i\hbar r\frac{\partial}{\partial r} + i\vec{\sigma}\cdot\vec{L}\right]$$

$$\vec{\sigma}\cdot\vec{L} = \left(\vec{L} + \frac{\vec{\sigma}}{2}\right)^2 - \vec{L}^2 - \left(\frac{\vec{\sigma}}{2}\right)^2 = j(j+1) - l(l+1) - 3/4$$

For $y_{j_+ = l+1/2,\, l,\, m_j} = \begin{pmatrix} \sqrt{\frac{l+m+1}{2l+1}}\, y_{lm} \\ \sqrt{\frac{l-m}{2l+1}}\, y_{l,m+1} \end{pmatrix}$,

$$y_{j_- = l-1/2,\, l,\, m_j} = \begin{pmatrix} -\sqrt{\frac{l-m}{2l+1}}\, y_{lm} \\ \sqrt{\frac{l+m+1}{2l+1}}\, y_{l,m+1} \end{pmatrix}.$$

$$(\vec{\sigma}\cdot\vec{L})\, y_{j_+ = l+1/2,\, l,\, m_j} = (l+3/2)(l+1/2) - l(l+1) - 3/4 = l\hbar$$

$$(\vec{\sigma}\cdot\vec{L})\, y_{j_- = l-1/2,\, l,\, m_j} = (l+1/2)(l-1/2) - l(l+1) - 3/4 = -(l+1)\hbar$$

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Hence $\vec{\sigma}\cdot\vec{p}\left(r^l Y_{j_+,l,m_j}\right) = \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[-i\hbar r\frac{\partial}{\partial r} + i\vec{\sigma}\cdot\vec{L}\right]\left[r^l Y_{j_+}\right]$

$= \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[-i\hbar l + i\hbar l\right]\left[r^l Y_{j+1}\right] = 0$

① Hence, $f_{j j_z} = r^l Y_{j_+ = l+1/2,\, l, m_j}$ ← polynomial (多项式)

$\rightarrow f_{j j_z} = \sqrt{\frac{l+m+1}{2l+1}}\, r^l Y_{lm} + j\sqrt{\frac{l-m}{2l+1}}\, r^l Y_{l,m+1}$.

Example: $f_{3/2\,3/2} = \begin{pmatrix} x+iy \\ 0 \end{pmatrix} \rightarrow f_{11} = x+iy$

Ⓐ

$rY_{11} = -\sqrt{\frac{3}{8\pi}}(x+iy)$

$rY_{10} = \sqrt{\frac{3}{4\pi}}z$

$rY_{1-1} = \sqrt{\frac{3}{8\pi}}(x-iy)$

$f_{3/2\,1/2} = \begin{pmatrix} \sqrt{\frac{2}{3}}Y_{10} \\ \sqrt{\frac{1}{3}}Y_{11} \end{pmatrix} \propto \begin{pmatrix} 2z \\ -x-iy \end{pmatrix} \rightarrow f_{10} = 2z - jx + ky$

$f_{3/2\,-1/2} \propto \begin{pmatrix} x-iy \\ 2z \end{pmatrix} \rightarrow \bar{f}_{10} = f_{10}\, j = x - iy + 2jz$

$f_{3/2\,-3/2} = \begin{pmatrix} 0 \\ x-iy \end{pmatrix} \rightarrow \bar{f}_{11} = f_{11}\, j = jx + ky$

$Tf = -i\sigma_2 f^* \quad i.e. \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \rightarrow \begin{pmatrix} -f_2^* \\ f_1^* \end{pmatrix}.$ → quaternions

$f_{lm} \rightarrow \bar{f}_{lm} = f_{lm}$

Ⓑ $r^2 Y_{22} = \frac{1}{2}\sqrt{\frac{15}{8\pi}}(x+iy)^2$, $r^2 Y_{21} = -\sqrt{\frac{15}{8\pi}}(x+iy)z$,

$r^2 Y_{20} = \sqrt{\frac{5}{16\pi}}(2z^2 - x^2 - y^2)$ $r^2 Y_{2-1} = \sqrt{\frac{15}{8\pi}}(x-iy)z$. $r^2 Y_{2-2} = \frac{1}{2}\sqrt{\frac{15}{8\pi}}(x-iy)^2$

⇒ $f_{22} = x^2 - y^2 + 2ixy$, $f_{21} = -4xz - 4iyz + j(x^2-y^2) - k\,2xy$

$f_{20} = 2z^2 - x^2 - y^2 - j(2xz) + k(2yz)$, $\bar{f}_{2m} = f_{2m}\, j$