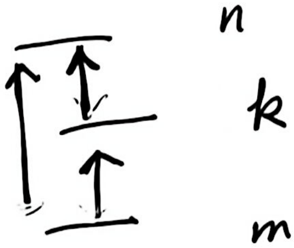
Lecture 2: Heisenberg's magic

§1: Reinterpretation of kinematic variable



$$x_{n,m}\, e^{i\omega(n,m)t}$$

$$= \begin{pmatrix} x_{11} e^{i\omega_{11}t} & x_{12} e^{i\omega_{12}t} & \cdots \\ x_{21} e^{i\omega_{21}t} & x_{22} e^{i\omega_{22}t} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$(x_1^2)_{nm} = \sum_k (X)_{nk} (X)_{km}$$

$$\omega_{nm} = \omega_{nk} + \omega_{km}$$

Ritz combination law

$$\ddot{x} + f(x) = 0 \leftarrow \text{in the matrix sense}$$

§ Quantization

$$\frac{1}{2\pi}\oint p\,dq = n\hbar \longleftrightarrow \boxed{\frac{\hbar}{2m} = \sum_{\alpha=0}^{\infty} \{\omega(n+\alpha,n)\,|x(n+\alpha,n)|^2 - \omega(n,n-\alpha)\,|X(n,n-\alpha)|^2\}}$$

§ Heinsenberg's zero point motion

$$[(\dot{x})^2]_{nn} = \frac{\hbar\omega}{m}(n+1/2)$$

§ Born - Jordan's matrix mechanics

$$[x, p] = i\hbar \leftarrow \text{Born's tomb stone}$$

Heisenberg's breakthough

① Interprete electron coordinate x as the relation between initial and final states

↙ light emission

consider a transition $n \to n-l$ ($l>0$), x is represented

$$x_{n-l,n}(t) = x_{n-l,n}\, e^{-i\omega_{n-l\leftarrow n} t}$$

according to EM, electron oscilation frequency is the light frequency. Consider the classic expansion

$$x_n(t) = \sum_l x_n(l)\, e^{-il\omega_n t}$$

we can make a correspondance $\boxed{x_{n-l,n}\, e^{-i\omega_{n-l\leftarrow n} t} \leftrightarrow x_n(l)\, e^{-il\omega_n t}}$

For $l=0$, $x_{n,n}$ corresponds to the zero frequency part of $x_n(t)$.

hence $x_{n,n}$ should be real.

Consider the light absorption, i.e $l<0$

$$\boxed{x_{n,n-|l|}\, e^{-i\omega_{n\leftarrow n-|l|} t} \leftrightarrow x_n(-|l|)\, e^{i|l|\omega_n t}}$$

Note $\omega_{n\leftarrow n-l} = -\omega_{n-l\leftarrow n}$, but not $\omega_{n+l\leftarrow n} = -\omega_{n-l\leftarrow n}$

typically not true

Classical relation: $x_n(-l) = x_n^*(l)$

Correspondingly, $x_{n,n-l} = x^*_{n-l,n}$, or $\boxed{x_{nm} = x^*_{mn}.}$

$$X_{mn}(t)\, e^{-i\omega_{m\leftarrow n}t} \leftrightarrow \begin{pmatrix} X_{11} & X_{12}\, e^{-i\omega_{1\leftarrow 2}t} & \cdots \\ X_{21}\, e^{-i\omega_{2\leftarrow 1}t} & X_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

Reinterpretation of kinematic coordniate

② Reconstruct the relation x^2 and x

$$(X^2)_{mn}\, e^{-i\omega_{m\leftarrow n}t} = \sum_k X_{m,k}\, e^{-i\omega_{m\leftarrow k}t}\, X_{k,n}\, e^{-i\omega_{k\leftarrow n}t}$$

$$= \sum_k X_{mk} X_{kn}\, e^{-i\omega_{m\leftarrow n}t}$$

↖ Ritz combination

↗ matrix product

Reinvention of matrix ! $x \to x^2 \to x^n$

③ reintepretation of equation of motion

Heinsenberg thought the Hamilton Eq should be valid.

We should maintain the form of dynamic Eq, but use the matrix interpretation

$$\ddot{x}_{mn} + [f(x)]_{mn} = 0$$

For the harmonic oscillator, $\boxed{\ddot{X}_{mn} + \omega_0^2 X_{mn} = 0.}$

4 reinterpretation of Bohr quantization

$$\oint p\,dx = nh \quad \Rightarrow m\oint \dot{x}^2\,dt = nh$$

$$x_n(t) = \sum_{l=-\infty}^{\infty} x_n(l)\, e^{-il\omega_n t} \quad \rightarrow \quad \dot{x}_n(t) = \sum_l (-il\omega_n)\, x_n(l)\, e^{-il\omega_n t}$$

$$\dot{x}^2 = \sum_l l^2\omega_n^2\, |x_n(l)|^2 + \sum_{l\neq l'} l l' \omega_n^2\, x_n(l)\, x_n^*(l')\, e^{-i(l-l')\omega_n t}$$

$$m\oint \dot{x}^2\,dt = m\sum_l l^2\omega_n^2\, |x_n(l)|^2 \oint dt \quad \leftarrow \quad T = \frac{2\pi}{\omega_n}$$

$$= \boxed{2\pi m \sum_l l^2 \omega_n\, |x_n(l)|^2 = nh}$$

Consider the limit $n \to \infty$, and the harmonic order $|l| \ll n$.

we view "n" as continous, and take derivative

$$\frac{\hbar}{m} = \frac{d}{dn}\sum_l l^2\omega_n\, |x_n(l)|^2 = \sum_l l\, \frac{d}{dn}\left[l\omega_n\, |x_n(l)|^2 \right]$$

since $l \ll n$, $l\frac{d}{dn} \rightarrow$ difference

$$l\frac{d}{dn}\left[l\omega_n\, |x_n(l)|^2 \right] = l\,\omega_{n+l}\, |x_{n+l}(l)|^2 - l\omega_n\, |x_n(l)|^2$$

$$\Rightarrow \quad \frac{\hbar}{m} = \sum_{l=-\infty}^{\infty} \Big[\omega_{n\leftarrow n+l}\, |\overset{X_{n,n+l}}{}|^2 - \omega_{n-l\leftarrow n}\, |\underset{X_{n-l,n}}{}|^2 \Big]$$

$$\omega_{n\leftarrow n+l} = -\omega_{n,n-l}\,, \qquad |X_{n-l,n}|^2 = |X_{n,n-l}|^2$$

and $\omega_{n\leftarrow n} = 0$

$$l<0 \text{ part}: \quad \omega_{n\leftarrow n-|l|}\, |X_{n,n-|l|}|^2 - \omega_{n+|l|\leftarrow n}\, |X_{n+|l|,n}|^2$$

$$= \omega_{n\leftarrow n+|l|}\, |X_{n+|l|,n}|^2 - \omega_{n-|l|\leftarrow n}\, |X_{n,n-|l|}|^2$$

we combine and simplify $\Rightarrow$

$$\boxed{\frac{\hbar}{2m} = \sum_{l=0}^{\infty} \left(\omega_{n\leftarrow n+l} \, |X_{n+l,n}|^2 - \omega_{n-l\leftarrow n} \, |X_{n,n-l}|^2 \right)}$$

f - sum rule, which is equivalent to $[x, p] = i\hbar$ later by Born and Jordan.

⊛ Application — harmonic oscillator

$$\ddot{x}_{n+l,n}(t) + \omega_0^2 \, x_{n+l,n}(t) = 0 \Rightarrow \quad X_{n+l,n}(t) = X_{n+l,n} e^{-i\omega_{n+l\leftarrow n} t}$$

$$\Rightarrow (\omega_0^2 - \omega_{n\leftarrow n\pm l}^2) \, X_{n,n\pm l} = 0$$

For non-vanishing matrix elements, we should have $\omega_{n\leftarrow n\pm l} = \pm \omega_0$.

For 1d motion, energy increases with action, i.e. there is at most one "l", such that $X_{n,n\pm l} \neq 0$. If there exsit two, $\omega_0 = \omega_{n\leftarrow n+l} = \omega_{n\leftarrow n+l'}$, then $l' = l$.

In the limit $n \to +\infty$, according to the correspondence principle

$$\omega_{n\leftarrow n\pm 1} = \pm \omega_0, \Rightarrow X_{n,n\pm 1} \text{ can be nonzero.}$$

For harmonic oscillator, frequency is independent on amplitude $\partial E/\partial J = \omega_0$, hence, $\omega_{n\leftarrow n\pm 1} = \pm \omega_0$ should be maintained even at n is small.

Then the f-sum rule:

$$\frac{\hbar}{2m} = \omega_0\left[|X_{n+1,n}|^2 - |X_{n,n-1}|^2\right]$$

$$\frac{\hbar}{2m} = \omega_0\left[|X_{n,n-1}|^2 - |X_{n-1,n-2}|^2\right]$$

$$\vdots$$

$$\frac{\hbar}{2m} = \omega_0\left[|X_{1,0}|^2 \quad\right]$$

since all the $|X_{n+1,n}|^2$ are positive, this sequence must be bound from below. we denote $n \geq 0$.

$$\Rightarrow |X_{1,0}|^2 = \frac{\hbar}{2m\omega_0}, \qquad |X_{n+1,n}|^2 = \frac{\hbar}{2m\omega_0}(n+1)$$

For the moment, we assume that $X_{n+1,n}$ is real

$$\Rightarrow \begin{cases} X_{n+1,n} = \sqrt{\frac{\hbar}{2m\omega_0}(n+1)} \\ X_{n-1,n} = \sqrt{\frac{\hbar}{2m\omega_0}n} \quad (n \geq 1) \end{cases}$$

$$X_{n+1,n}(t) = X_{n+1,n}\, e^{i\omega_0 t}$$

$$X_{n-1,n}(t) = X_{n-1,n}\, e^{-i\omega_0 t}$$

Zero point motion

$$\oint p\, dq = m\int_0^T (\dot{x})^2\, dt$$

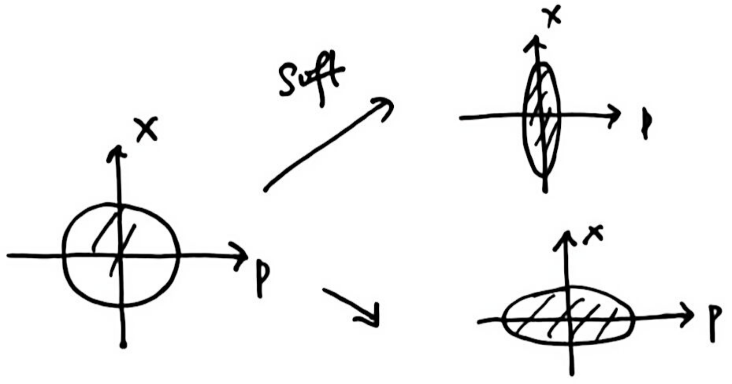
$$(\dot{x}^2)_{nn} = \sum \dot{x}_{nm}\dot{x}_{mn} = \dot{x}_{n,n-1}\dot{x}_{n-1,n} + \dot{x}_{n,n+1}\dot{x}_{n+1,n}$$

$$X_{n,n-1} = \left(\frac{n\hbar}{2m\omega_0}\right)^{1/2} e^{i\omega_0 t} \Rightarrow \dot{X}_{n,n-1} = i\omega_0\left(\frac{n\hbar}{2m\omega_0}\right)^{1/2} e^{i\omega_0 t}$$

$$(\dot{x}^2)_{nn} = \omega_0^2\left[\frac{n\hbar}{2m\omega_0} + \frac{(n+1)\hbar}{2m\omega_0}\right] = \frac{\hbar\omega_0}{m}(n+1/2)$$

$$J = \left(\oint p\,dq\right)_{nn} = \hbar\omega_0(n+1/2)\int_0^T dt = 2\pi\hbar(n+1/2) = (n+1/2)h$$

Since $\omega = \frac{\partial E}{\partial J}$ $\Rightarrow$ $\boxed{E = (n+1/2)\hbar\omega}$ ← $1/2$: zero point motion



$\boxed{\Delta x \cdot \Delta p \geq \frac{\hbar}{2}}$

(x) Born-Jordan commutation

x, p donot commute if they are matrices, $xp - px = m(x\dot{x} - \dot{x}x)$

Check diagonal part $(x\dot{x} - \dot{x}x)_{nn}$,

$$\sum_{\ell} x_{n,n-\ell}\,\dot{x}_{n-\ell,n} - \dot{x}_{n,n-\ell}\,x_{n-\ell,n}$$

$$= \sum_{\ell} |x_{n,n-\ell}|^2(-i\omega_{n-\ell\leftarrow n}) - (-i\omega_{n\leftarrow n-\ell})|x_{n-\ell,n}|^2$$

if $\ell > 0$, $\sum_{\ell>0} -2i\omega_{n-\ell\leftarrow n}\,|x_{n,n-\ell}|^2$

if $\ell < 0$, $\sum_{|\ell|>0} |x_{n,n+|\ell|}|^2(-2i\,\omega_{n+|\ell|\leftarrow n})$

$$\Rightarrow (x\dot{x} - \dot{x}x)_{nn} = +2i\sum_{\ell>0}\left[\omega_{n+\ell\leftarrow n}\,|x_{n+\ell,n}|^2 - \omega_{n-\ell\leftarrow n}\,|x_{n,n-\ell}|^2\right]$$

$$= \frac{\hbar}{m}i$$

$$\Rightarrow \boxed{(xp - px)_{nn} = i\hbar}$$