Lecture 3 The birth of Matrix mechnics

Born, Jordan 1925

Dirac, P A M 1925

$[p, x] = \frac{\hbar}{i}$ ← Born's tombstone

$$\left.\begin{aligned} \dot{x} &= \frac{1}{i\hbar}[x, H] \\ \dot{p} &= \frac{1}{i\hbar}[p, H] \end{aligned}\right\} \text{ Born and Jordan}$$

$$\frac{1}{i\hbar}[x, p] \leftrightarrow \{x, p\}$$

$\frac{1}{i\hbar}[f_1, f_2] \leftrightarrow \{f_1, f_2\}$ ← Dirac canonical quantization

§1 Hermitian matrix $\longleftrightarrow$ mechanical quantity

Post 1: Coordinate and momentum should be represented as matrix $p_{m,n}\, e^{-i\omega_{m\leftarrow n}t}$, $x_{mn}\, e^{-i\omega_{m\leftarrow n}t}$.

When we define matrix product, say $(x\,p)_{mn}\, e^{-i\omega_{m\leftarrow n}t}$

$$= \sum_k x_{mk}\, p_{kn}\, e^{-i(\omega_{m\leftarrow k}+\omega_{k\leftarrow n})t}$$

Hence. Post 1 really relys on the Ritz combination rule.

(x, p) forms the complete set of mechanical observables,

$O(p,x)$ in principle can be expressed power series of x and p. We write down the classic expression

$$O = \sum_l O_n(l)\, e^{-il\omega_n t}.$$

The correspondence between classic and quantum version

$$O_n(l)\, e^{-il\omega_n t} \longleftrightarrow O_{n-l,n}\, e^{-i\omega_{n-l\leftarrow n}t}$$

向下兼容

for l>0

注意规定

but $O_{n+l,n}\, e^{il\omega_{n\leftarrow n+l}t}$

$$= O^*_{n,n+l}\left(e^{-il\omega_{n\leftarrow n+l}t}\right)^*$$

Post 2. frequency recombination

$$\omega_{j\leftarrow k} + \omega_{k\leftarrow l} + \omega_{l\leftarrow j} = 0$$

At this stage, we can relate $\hbar\omega_{m\leftarrow n} = W_n - W_m$.
But whether W is energy or not, we will need to prove.

Post 3. Hamiltonian Eq is the same as before, but should be intepreted in terms of the matrix language.

$$H = \frac{p^2}{2m} + V(q)$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Dynamical rules are the same but the interpretation of kinematics (meaning of q and p) needs to be redone.

§ Establishment of the canonical commutation rule

> Post 4: The diagonal matrix elements of
> $$(px-xp)_{nn} = \hbar/i .$$

we cannot prove, but show its equivalente to the f-sum rule.

$$(px-xp)_{nn} = m(\dot{x}x - x\dot{x})_{nn} = m\sum_k \dot{X}_{nk} X_{kn} - X_{nk}\dot{X}_{kn}$$

$$\dot{X}_{nk} = -i\omega_{n\leftarrow k} X_{nk} , \quad \dot{X}_{kn} = -i\omega_{k\leftarrow n} X_{kn} .$$

$$(px-xp)_{nn} = -i\sum_k \omega_{n\leftarrow k} X_{nk} X_{kn} - \omega_{k\leftarrow n} X_{nk} X_{kn}$$

① Set $k = n+l$, for $l>0$

$$m\sum_{l>0} -i\omega_{n\leftarrow n+l} |X_{n,n+l}|^2 + i\omega_{n+l\leftarrow n} |X_{n+l,n}|^2$$

$$= -2mi\sum_{l>0} \omega_{n\leftarrow n+l} |X_{n+l,n}|^2$$

② if $l<0$, then $k = n-|l|$,

$$(px-xp)_{nn} = -2im\sum \omega_{n\leftarrow n-|l|} X_{n,n-|l|}, X_{n-|l|,n}$$

$$= 2im \sum \omega_{n-|l|,n} |X_{n,n-|l|}|^2$$

$$\Rightarrow (px - xp)_{nn} = 2im \sum_{l>0} \omega_{n\leftarrow n+l}\left[|X_{n+l,n}|^2 - \omega_{n-l\leftarrow n} |X_{n,n-l}|^2 \right]$$

by using the f-sum rule

$$(px - xp)_{nn} = 2im \frac{\hbar}{2m} = \hbar/i .$$

Hence, we show that Heisenberg's quantization rule is equivalent to the diagonal matrix element $[p, x] = \hbar/i$.

* Jordan's contribution: proof of $([p.x])_{m,n} = 0$ if $m \neq n$.

Basically, Jodan proved $\frac{d}{dt}([p,x])_{m,n} = 0$, if the off diagonal matrix element is non-zero, then it would have time dependence $[p,x]_{mn} e^{-i\omega_{mn}t}$.

We consider a simple case that $H = H_1(p) + H_2(x)$

Define $g = \frac{i}{\hbar}[p, x]$.

Then $$\frac{dg}{dt} = \frac{i}{p}\left[\dot{p}x + p\dot{x} - \dot{x}p - x\dot{p} \right]$$

$$\dot{g} = \frac{i}{\hbar}\left(- \frac{\partial H_2(x)}{\partial x} x + x \frac{\partial H_2(x)}{\partial x} - \frac{\partial H_1(p)}{\partial p} p + p \frac{\partial H_1(p)}{\partial p} \right)$$ (5)

Since H_2 only depends on x, we have $x \frac{\partial H_2(x)}{\partial x} = \frac{\partial H_2(x)}{\partial x} x$,

then $\dot{g} = 0$, then the off-diagonal matrix element

of $g_{m,n} = 0 \Rightarrow g = 1$, i.e $\boxed{[p, x] = \hbar/i}$

If H contains term mixing p, x, the proof is more complicated. Nevertheless, it can be done. We will not show the details here.

(*) With cannoical commutation rule, we can express $\dot{p} = \frac{1}{i\hbar}[p, H]$, $\dot{x} = \frac{1}{i\hbar}[x, H]$

Again for simplicity, we prove the case $H = H_1(p) + H_2(x)$.

Check $\dot{p} = -\frac{\partial}{\partial x} H_2(x)$, assume $H_2(x) = \sum_n a_n x^n$

$\Rightarrow \dot{p} = -\sum_n a_n n x^{n-1}$.

It's easy to show $[p, x^n] = \sum_{i=0}^{n-1} x^i [p, x] x^{n-i-1}$

$= -i\hbar x^{n-1}$

$\Rightarrow \dot{p} = \frac{1}{i\hbar}[p, H]$.

Similarly, we have $\dot{x} = \frac{1}{i\hbar}[x, H]$.

If $\dot{g}_1 = \frac{1}{i\hbar}[g_1, H]$, $\dot{g}_2 = \frac{1}{i\hbar}[g_2 H]$, then

$$\frac{d}{dt}(g_1 g_2) = \dot{g}_1 g_2 + g_1 \dot{g}_2 = \frac{1}{i\hbar}\left([g_1 H] g_2 + g_1 [g_2 H]\right)$$

$$= \frac{1}{i\hbar}[g_1 g_2, H]$$

Since O in principle could be expanded in terms of power series of p, x, then $\dot{O} = \frac{1}{i\hbar}[O, H]$.

⊛ if $H(p,x)$ doesnot depend on t explicitly, then

$\frac{d}{dt} H = 0$. $\Rightarrow$ H is diagonal since its off diagonal matrix elements are time-depedent if it is non-zero.

Post 5: $H_{nn} = E_n$, i.e the diagonal matrix etements of H is the energy of the stationay state E_n.

$\because$ $\dot{O} = \frac{1}{i\hbar}[O, H]$, $\qquad \dot{O}_{mn} = \frac{1}{i\hbar}(O_{mn} H_{nn} - H_{mm} O_{mn})$

$$-i\omega_{m\leftarrow n} O_{mn} = \frac{1}{i\hbar} O_{mn}(E_n - E_m)$$

$$\boxed{\hbar\omega_{m\leftarrow n} = E_n - E_m}$$

§ Canonical quantization

$$\{f_1, f_2\}_{p,x} = -\frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial p}$$ - Poisson bracket

Poisson bracket does not depend on the choice of coordinate and momentum. We can also use (J, θ) to define Poisson bracket

$$\{f_1, f_2\} = \{f_1, f_2\}_{p,x} = \{f_1, f_2\}_{J,\theta} = -\frac{\partial f_1}{\partial J}\frac{\partial f_2}{\partial \theta} + \frac{\partial f_1}{\partial \theta}\frac{\partial f_2}{\partial J}$$

Classic mechanics	QM equation of motion
$\dot{p} = \{p, H\}$	$\dot{p} = \frac{1}{i\hbar}[p, H]$
$\dot{q} = \{q, H\}$	$\dot{q} = \frac{1}{i\hbar}[q, H]$

Dirac made the conjecture that

$$\{\ \} \leftrightarrow \frac{1}{i\hbar}[\ ].$$

Below we will use p and x to check the correspondence between $\{p, x\} \leftrightarrow \frac{1}{i\hbar}[p, x]$.

Consider the matrix element $[p, x]_{n,n-\ell}$ where n corresponds to the classic action $J = n\hbar$, and the classic variable $\theta = \omega_n t$, and $\omega_n = \partial E/\partial J\,\big|_{J=n\hbar}$.

Let's temporally suppress the time dependence

$$[p, x]_{n,n-l} = (px - xp)_{n,n-l}$$

$$= \sum_k (P_{n,n-k} - P_{n-(l-k),n-l})\, X_{n-k,n-l}$$

$$- \sum_k (X_{n,n-(l-k)} - X_{n-k,n-l})\, P_{n-(l-k),n-l}$$

matrix elements

$$= \sum_k \hbar (l-k) \frac{\partial}{\partial J} P_{n,n-k}\, X_{n-k,n-l} - k\hbar \frac{\partial}{\partial J} X_{n,n-(l-k)}\, P_{n-(l-k),n-l}$$

both $n \to n-(l-k)$ $\qquad$ $n \to n-k$

↓ restore time dependence

$$\frac{i}{\hbar}[p, x]_{n,n-l}\, e^{-i\omega_{n\leftarrow n-l} t} = \sum_k \frac{\partial}{\partial J}\left(P_{n,n-k}\, e^{-i\omega_{n\leftarrow n-k} t}\right) \frac{\partial}{\partial \theta}\left(X_{n-k,n-l}\, e^{-i\omega_{n-k \leftarrow n-l} t}\right)$$

$$- \frac{\partial}{\partial J}\left(X_{n,n-(l-k)}\, e^{-i\omega_{n\leftarrow n-(l-k)} t}\right) \frac{\partial}{\partial \theta}\left(P_{n-(l-k),n-l}\, e^{-i\omega_{n-(l-k)\leftarrow n-l} t}\right)$$

check $-i\omega_{n-k \leftarrow n-l} \cdot t \to -i(k-l)\omega_n t = i(l-k)\theta$

$$-i\omega_{n-(l-k)\leftarrow n-l} \cdot t \to ik\theta$$

According to the correspondence

$$(l>0) \quad O_{n-l,n}\, e^{-i\omega_{n-l\leftarrow n} \cdot t} \leftrightarrow O_n(l)\, e^{-il\omega_n t}$$

$$O_{n,n-l}\, e^{-i\omega_{n\leftarrow n-l} t} \leftrightarrow O_n(-l)\, e^{il\omega_n t}$$

for $k>0$, $P_{n,n-k} \to P_n(-k)$

<0 $\quad P_{n,n-k} = P_{n,n+|k|} \to P_{n-k}(-k) \simeq P_n(-k)$

at $n \gg k$

$X_{n-k,n-l} \to X_n(-l+k)$ → the difference between two indices

sum over the left and right indices

and take the leading order

$$X_{n,n-(l-k)} \to X_n(-l+k)$$

$$P_{n-(l-k),n-l} \to P_n(-k)$$

$$\Rightarrow \frac{i}{\hbar}[P,x]_{n,n-l}\, e^{-i\omega_{n\leftarrow n-l}t} = \sum_k \cdots\cdots$$

$$\to \sum_k \frac{\partial}{\partial J}\left(P_n(-k)\, e^{ik\omega_n t}\right)\frac{\partial}{\partial \theta}\left(X_n(-l+k)\, e^{-i(-l+k)\omega_n t}\right)$$

$$- \frac{\partial}{\partial J}\left(X_n(-l+k)\, e^{i(l-k)\omega_n t}\right)\frac{\partial}{\partial \theta}\left(P_n(-k)\, e^{ik\omega_n t}\right)$$

$$= O_n(-l)\, e^{il\omega_n t}$$

where $O = \frac{\partial}{\partial J}P\frac{\partial}{\partial \theta}X - \frac{\partial}{\partial J}x\frac{\partial}{\partial \theta}P = -\{P,x\}$

Check

$$O_n(t) = \frac{\partial}{\partial J}P_n(t)\frac{\partial}{\partial \theta}X_n(t) - \frac{\partial}{\partial J}X_n(t)\frac{\partial}{\partial \theta}P_n(t)$$

$$O_n(-l) = \int dt \; e^{-il\omega_n t} \; O_n(t)$$

$$P_n(t) = \sum_k e^{ik\omega_n t} \; P_n(-k)$$

$$X_n(t) = \sum_k e^{i(l-k)\omega_n t} \; X_n(-l+k)$$

$$O_n(-l) = \int dt \; e^{-il\omega_n t} \Big\{ \frac{\partial}{\partial J} \sum_k (e^{ik\omega_n t} P_n(-k) \frac{\partial}{\partial \theta} \sum_{k'} e^{i(l-k')\omega_n t} X_n(-l+k') - \dots \Big\} \quad (\text{Pick } k = k')$$

$$\sim e^{-il\omega_n t} \Big(\frac{\partial}{\partial J} \sum_k e^{ik\omega_n t} P_n(-k) \frac{\partial}{\partial \theta} \sum_k e^{i(l-k)\omega_n t} X_n(-l+k) - \dots \Big)$$

~ all the t-dependent term cancels.

In principle, we can apply to arbitany f_1, f_2

$$\left(\frac{1}{i\hbar} [f_1 \, f_2] \right)_{n,n-l} \rightarrow \{f_1, f_2\}_n (-l) \; e^{il\omega_n t}$$

$$\frac{1}{i\hbar} [f_1 \, f_2] \longleftrightarrow \{f_1, f_2\}$$

Canonical quantization