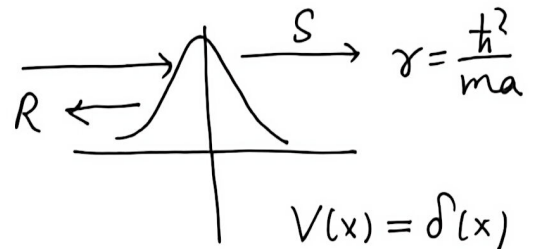
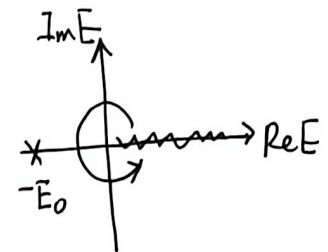


Lecture 6 1D Scattering - bound states

{ δ -potential



$$S = \frac{1}{1 + i/k a} \quad R = \frac{-i/k a}{1 + i/k a}$$



{ Analytic continuation $k = \sqrt{2mE}/\hbar$

Attractive- δ potential $S(E) = \frac{\sqrt{E/E_0}}{\sqrt{E/E_0} - i} \quad \left\{ \begin{array}{l} E_0 = \frac{\hbar^2 \beta^2}{2m} \\ \beta = |a|^{-1} \end{array} \right.$

pole: $\sqrt{E/E_0} = i \Rightarrow E = -E_0$

bound state $\Leftrightarrow$ pole of scattering amplitude on the physical sheet.

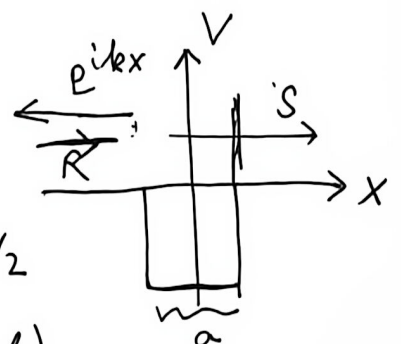
{ 1D attractive potential well at least contains one bound state.

{ Finite-width potential well - scattering amplitude v.s. bound states

{ perfect transmission — $E_n = \frac{\hbar^2}{2m} \left(\frac{n\pi}{a} \right)^2 - |V_0|$

{ Breit-Wigner $E \approx E_n$

$$S(E) \approx \pm \frac{i \Gamma_n/2}{(E - E_n) + i \Gamma_n/2}$$

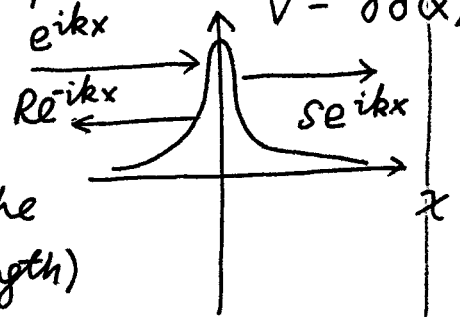


$E = E_n - i \Gamma_n/2$ (2nd Riemann sheet)

We first consider the simplest case of δ -potential

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi = (E - \gamma \delta(x)) \psi$$

define $k = \sqrt{\frac{2mE}{\hbar^2}}$, and $\gamma = \frac{\hbar^2}{ma}$ (a carries the unit of length)



$$\Rightarrow \frac{d^2}{dx^2} \psi + k^2 \psi = \frac{\gamma}{a} \delta(x) \psi.$$

consider the boundary condition under incident/transmission/reflection

$$\psi(x) = \begin{cases} e^{ikx} + R e^{-ikx} & x < 0 \\ S e^{ikx} & x > 0 \end{cases}$$

Q: Should $\psi(x)$ and $\psi'(x)$ be continuous at $x=0$?

A: $\psi(x)$ should be continuous but $\psi'(x)$ not

$$\int_{0^-}^{0^+} dx \frac{d^2}{dx^2} \psi + k^2 \psi = \frac{\gamma}{a} \int_{0^-}^{0^+} dx \delta(x) \psi(x)$$

$$\psi'(0^+) - \psi'(0^-) = \frac{\gamma \psi(0)}{a}$$

$$\Rightarrow \begin{cases} 1 + R = S \\ ikS - ik(1 - R) = \frac{\gamma S}{a} \end{cases}$$

$\Rightarrow$

$$\begin{cases} S = \frac{1}{1 + i\gamma/a} \\ R = \frac{-i\gamma/a}{1 + i\gamma/a} \end{cases}$$

$|S|^2$ transmission coefficient
 $|R|^2$ reflection coefficient!

Ex: ① check that $|S|^2 + |R|^2 = 1$

② although $\psi'(x)$ is not continuous at $x=0$, but the current

density $j_x = \frac{\hbar}{2m} \left[\psi^*(x) \frac{d}{dx} \psi(x) - \psi(x) \frac{d}{dx} \psi^*(x) \right]$ remains

continuous at $x=0$.

Comment: ① at low energy limit $k \rightarrow 0 \Rightarrow \begin{cases} S = -ika \rightarrow 0 \\ R = -1 \end{cases}$
complete reflection.

$k \rightarrow \infty$ (high energy) $\Rightarrow \begin{cases} S = 1 \\ R = \frac{-i}{ka} \rightarrow 0 \end{cases}$ complete transmission.

ANALYTICAL CONTINUATION

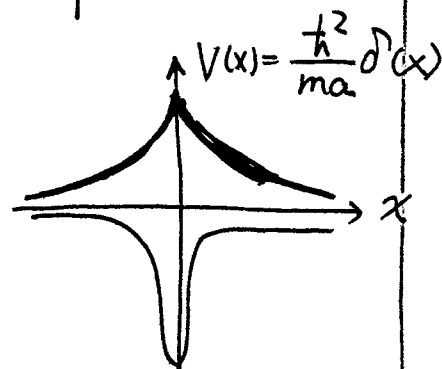
define $E_0 = \frac{\hbar^2}{2ma^2}$, we write S, R as

$$\begin{cases} S = \frac{\sqrt{E/E_0}}{\sqrt{E/E_0} \pm i} \\ R = \frac{\mp i}{\sqrt{E/E_0} \pm i} \end{cases}$$

" $\pm$ " refers to the case $a > 0$ and $a < 0$, respectively.

a. For the case of $a > 0$, we know that $E > 0$, there's no ambiguity for the interpretation of $\sqrt{E}$, and S, R are regular with respect to E .

b. For the case of $a < 0$, we know that in addition to the scattering states, which can be described by switching the sign of a , we also have bound



state.

Solution

for bound state:

choosing $\psi = \begin{cases} e^{\beta x}, & x < 0 \\ e^{-\beta x}, & x > 0 \end{cases}$

$$E = -\frac{\hbar^2 \beta^2}{2m} < 0.$$

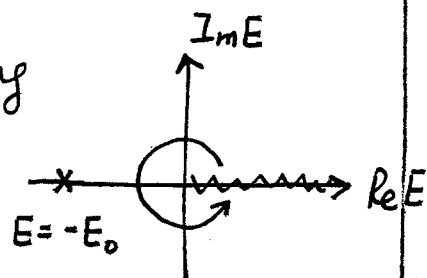
according to continuity relation, $2\beta = \frac{2}{a}$

$$\psi'(0^+) - \psi'(0^-) = \frac{2\psi(0)}{a} \Rightarrow \text{or } \beta = \frac{1}{a}.$$

The question is "Do we have a unified description for both scattering and bound states?"

The answer is yes! we need do analytic continuation of $S(E)$ and $R(E)$ for $E < 0$.

We need to define the branch cut of $\sqrt{E}$: along the positive E -axis, i.e. if we write $E = |E|e^{i\theta}$ with $0 \leq \theta < 2\pi$, we define $\sqrt{E} = \sqrt{|E|} e^{i\theta/2}$.



This Riemann sheet is the physical sheet, and the other one $\sqrt{E} = -\sqrt{|E|} e^{i\theta/2}$ is called the un-physical sheet, or the second Riemann sheet.

Now, we can see the bound state corresponds to the simple pole on the physical sheet.

$$\begin{cases} S = \frac{\sqrt{E/E_0}}{\sqrt{E/E_0} - i} \\ R = \frac{i}{\sqrt{E/E_0} - i} \end{cases}; \quad \sqrt{E/E_0} \xrightarrow{E=-E_0} i, \text{ a simple pole.}$$

Because S and R diverge, we can maintain the "reflection" and "transmission" wave amplitude, but set the incident wave to zero, i.e. no incident wave, but we can have "reflection/transmission", because $\frac{S}{R} \rightarrow \infty$. Then

at $E = -E_0$, $R = S \rightarrow \infty$, we can write $\psi(x) = e^{-\beta|x|}$, which is just what we have directly solved.

Why we can do this? Look back at the Schrödinger Eq

$$\left(\frac{d^2}{dx^2} + \frac{2mE}{\hbar^2} \right) \psi = \frac{2}{a} \delta(x) \psi, \quad "E" \text{ is a parameter.}$$

At the level of differential Eq, we can treat E as complex. Once we

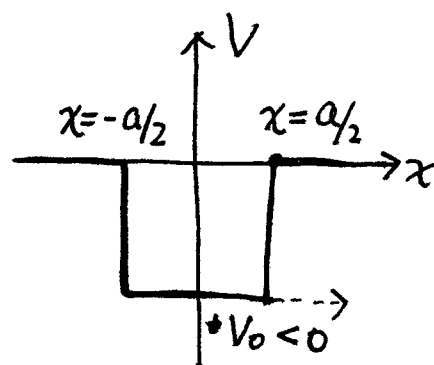
have solved the scattering states for $E > 0$, by complex continuation to $E < 0$.
 $\swarrow$
 S and R for

we arrive bound states.

★ Bound states correspond to poles of scattering amplitudes $\swarrow$ $S(E), R(E)$ on the physical sheet of E ! And scattering states correspond to a branch cut.

§3. potential well with finite depth

$$V(x) = \begin{cases} 0 & \text{for } |x| > \frac{a}{2} \\ V_0 \text{ (negative)} & \text{for } |x| < \frac{a}{2} \end{cases}$$



① we first look at the solution of bound states with $E < 0$.

define $k' = \sqrt{\frac{2m(E+|V_0|)}{\hbar^2}}$ and $\beta = \sqrt{\frac{2m|E|}{\hbar^2}}$.

The hamiltonian has parity symmetry, i.e. $H(x) = H(-x)$, or, $PHP^{-1} = H$.

The operation of parity transformation $P\psi(x) = \psi(-x)$. Since $[P, H] = 0$,

we can find H and P common eigenstates, i.e. eigensolutions that are even or odd functions with respect to x .

② even parity solution

$$\psi(x) = \begin{cases} A \cos k'x & , |x| \leq \frac{a}{2} \\ B e^{-\beta|x|} & |x| \geq \frac{a}{2} \end{cases}$$

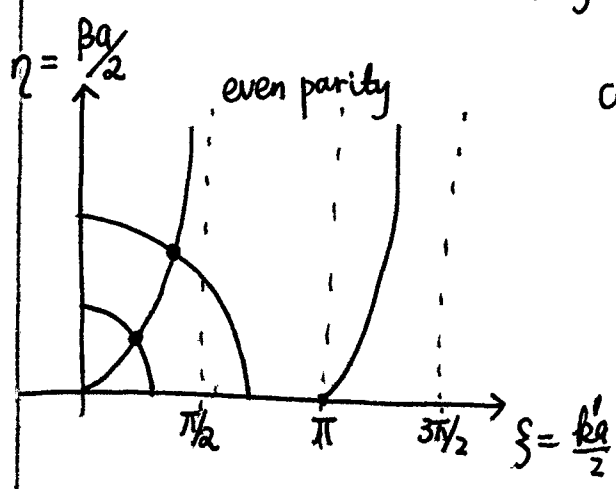
Ex: Based on the fact that the jumps of $V(x)$ at $x = \pm a/2$ are finite, prove that both $\psi(x)$ and $\psi'(x)$ are continuous at $x = \pm a/2$.

Since both ψ and ψ' are continuous, $(\ln \psi)' = \frac{\psi'}{\psi}$ should be continuous.

$$\frac{-k' \sin k'x}{\cos k'x} = \frac{-\beta e^{-\beta x}}{e^{-\beta x}} \Big|_{x=a/2} \Rightarrow k \tan ka/2 = \beta a$$

AMPAD

define $\begin{cases} \frac{ka}{2} = \xi > 0 \\ \frac{\beta a}{2} = \eta > 0 \end{cases} \Rightarrow \begin{cases} \xi^2 + \eta^2 = \frac{2m|V_0|}{\hbar^2} \cdot \frac{a^2}{4} = \frac{|V_0|}{\frac{\hbar^2}{2m(a/2)^2}} \\ \xi \tan \xi = \eta \end{cases}$

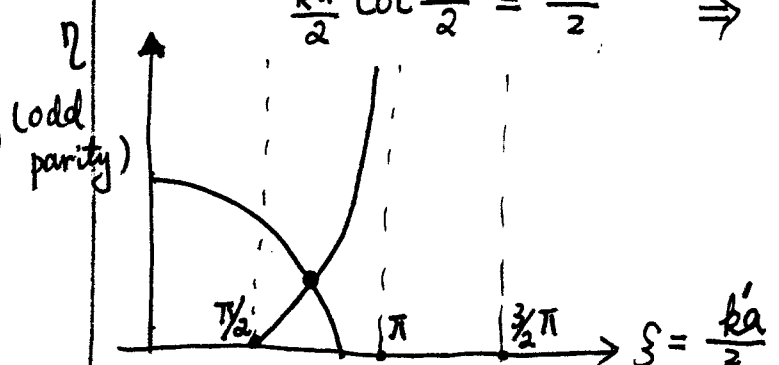


comment:
① no matter how small $|V_0|$ is, there's always a bound state solution in the even parity channel.

② odd parity solution $\psi(x) = \begin{cases} A \sin k'x, & |x| \leq a/2 \\ B e^{-\beta x} & x > a/2 \\ -B e^{-\beta x} & x < -a/2 \end{cases}$

$(\ln \psi)'$ continuity at $x = a/2 \Rightarrow \frac{k' \cos k'x}{\sin k'x} = -\beta \frac{e^{-\beta x}}{e^{-\beta x}} \Big|_{x=a/2}$

$$-\frac{ka}{2} \cot \frac{ka}{2} = \frac{\beta a}{2} \Rightarrow \begin{cases} \xi^2 + \eta^2 = \frac{|V_0|}{\frac{\hbar^2}{2m(a/2)^2}} \\ \eta = -\xi \cot \xi \end{cases}$$



Comment: The odd solution only appear when $\xi^2 + \eta^2 \geq \left(\frac{\pi}{2}\right)^2$, or

①

$$\frac{|V_0|}{\frac{\hbar^2}{2m(\frac{a}{2})^2}} \geq \left(\frac{\pi}{2}\right)^2 \Rightarrow |V_0| \geq \frac{\hbar^2 \pi^2}{2ma^2}.$$

② imagine that we gradually enlarge the potential depth $|V_0|$, as

$$\frac{|V_0|}{\frac{\hbar^2}{2m(\frac{a}{2})^2}} \sim \left(\frac{n\pi}{2}\right)^2, \text{ i.e. } |V_0| = \frac{\hbar^2 \pi^2}{2ma^2} n^2, \text{ a new } (n+1)\text{th bound state appears at zero energy.}$$

AMPAD

③ For infinite depth, i.e. $\xi^2 + \eta^2 \rightarrow \infty$,

the solutions appear at $\begin{cases} \xi = \frac{n\pi}{2}, \text{ or } k = \frac{n\pi}{a}. \\ \eta = \infty \end{cases} \leftarrow \text{wavefunction only appears inside well.}$

④ the ground state is "no-node",
n-th excited states has n-nodes.

2. Now we solve the scattering problem for $E > 0$.

$$\psi(x) = \begin{cases} e^{ikx} + R e^{-ikx} & x < -\frac{a}{2} \quad (\text{incident + reflection}) \\ A e^{ik'x} + B e^{-ik'x} & -\frac{a}{2} < x < \frac{a}{2} \\ S e^{ikx} & \frac{a}{2} < x \quad (\text{transmission}) \end{cases}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$, and $k' = \sqrt{\frac{2m(E+|V_0|)}{\hbar^2}}$.

at $x = -\frac{a}{2}$

$$\begin{cases} e^{-ik\frac{a}{2}} + R e^{ik\frac{a}{2}} = A e^{-ik'\frac{a}{2}} + B e^{ik'\frac{a}{2}} \\ ik e^{-ik\frac{a}{2}} + R(-ik) e^{ik\frac{a}{2}} = A(ik') e^{-ik'\frac{a}{2}} + B(-ik') e^{ik'\frac{a}{2}} \end{cases}$$

$$\Rightarrow \left. \begin{aligned} A e^{-ik'a/2} &= \frac{1}{2} \left(1 + \frac{k}{k'}\right) e^{-ik'a/2} + \frac{1}{2} R \left(1 - \frac{k}{k'}\right) e^{ik'a/2} \\ B e^{ik'a/2} &= \frac{1}{2} \left(1 - \frac{k}{k'}\right) e^{-ik'a/2} + \frac{1}{2} R \left(1 + \frac{k}{k'}\right) e^{ik'a/2} \end{aligned} \right\} \quad (1)$$

at $x = a/2$

$$\begin{cases} S e^{ik'a/2} = A e^{ik'a/2} + B e^{-ik'a/2} \\ ik S e^{ik'a/2} = A(ik') e^{ik'a/2} + B(-ik') e^{-ik'a/2} \end{cases}$$

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$$\Rightarrow (2) \begin{cases} A e^{ik'a/2} = \frac{1}{2} S \left(1 + \frac{k}{k'}\right) e^{ik'a/2} \\ B e^{-ik'a/2} = \frac{1}{2} S \left(1 - \frac{k}{k'}\right) e^{ik'a/2} \end{cases}$$

compare (1) and (2)

$$\left(1 + \frac{k}{k'}\right) + R \left(1 - \frac{k}{k'}\right) e^{ika} = S \left(1 + \frac{k}{k'}\right) e^{i(k-k')a}$$

$$\left(1 - \frac{k}{k'}\right) + R \left(1 + \frac{k}{k'}\right) e^{ika} = S \left(1 - \frac{k}{k'}\right) e^{i(k+k')a}$$

$$\Rightarrow \left. \begin{aligned} S &= e^{-ika} \frac{1}{\cos k'a - \frac{i}{2} \left(\frac{k}{k'} + \frac{k'}{k}\right) \sin k'a} \\ R &= e^{-ika} \frac{\frac{i}{2} \left(\frac{k'}{k} - \frac{k}{k'}\right) \sin k'a}{\cos k'a - \frac{i}{2} \left(\frac{k}{k'} + \frac{k'}{k}\right) \sin k'a} \end{aligned} \right\}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$

and $k' = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$

(★) Again let us do analytic continuation of the energy variable E . The pole is located

$$\cos k'a = \frac{i}{2} \left(\frac{k}{k'} + \frac{k'}{k}\right) \sin k'a$$

using $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \frac{1 - \tan^2 \frac{k'a}{2}}{\tan \frac{k'a}{2}} = \left(\cot \frac{k'a}{2} - \tan \frac{k'a}{2} \right) = \frac{ik}{k'} - \frac{k'}{ik}$$

the solution is equivalent to $\cot \frac{k'a}{2} = \frac{ik}{k'}$, or $-\tan \frac{k'a}{2} = \frac{ik}{k'}$.

according to the convention of using the first Riemann sheet

$$k = \sqrt{\frac{2mE}{\hbar}} = i\beta \Rightarrow \text{the above Eqs become}$$

$$\boxed{k' \cot \frac{k'a}{2} = -\beta, \text{ or } k' \tan \frac{k'a}{2} = \beta}$$

Which are just the energies we solved before for bound states.

AMPAD a) if $k' \tan \frac{k'a}{2} = \beta$ is satisfied, then

$$\sin k'a = \frac{2 \frac{\beta}{k'}}{1 + (\frac{\beta}{k'})^2} = \frac{2}{\frac{k'}{\beta} + \frac{\beta}{k'}}$$

$\frac{R}{S} = \frac{i}{2} \left(-\frac{i\beta}{k'} + \frac{k'}{i\beta} \right) \sin k'a = +1$, thus it corresponds to even parity.

b) if $k' \cot \frac{k'a}{2} = -\beta \Rightarrow \tan \frac{k'a}{2} = -\frac{k'}{\beta} \Rightarrow \sin k'a = \frac{-2 \frac{k'}{\beta}}{1 + (\frac{k'}{\beta})^2} = \frac{-2}{\frac{k'}{\beta} + \frac{\beta}{k'}}$
 $\Rightarrow R/S = -1$, thus it corresponds to odd parity solution.

§3: Transmission resonances

$$S(E) = e^{-ika} \frac{1}{\cos k'a - \frac{i}{2} \left(\frac{k'}{k} + \frac{k}{k'} \right) \sin k'a}$$

$$T(E) = |S(E)|^2 = \frac{1}{(\cos k'a)^2 + \frac{1}{4} \left(\frac{k'}{k} + \frac{k}{k'} \right)^2 \sin^2 k'a}$$

$$= \frac{1}{1 + \frac{1}{4} \left(\frac{k'}{k} - \frac{k}{k'} \right)^2 \sin^2 k'a}$$

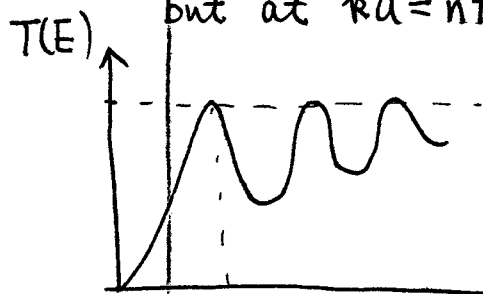
The perfect transmission occurs at $k'a = n\pi$.

$$\left(\frac{k}{k'} - \frac{k'}{k} \right)^2 = \left(\frac{k^2 - k'^2}{kk'} \right)^2 = \frac{|V_0|^2}{E(E+|V_0|)} = \frac{1}{E/|V_0|(E/|V_0|+1)}$$

$$\Rightarrow T(E) = \left[1 + \frac{\sin^2 k'a}{4 E/|V_0|(E/|V_0|+1)} \right]^{-1}$$

if $k'a \neq n\pi$, because V_0/E typically speaking $\gg 1$, $T(E) \sim E/V_0 \ll 1$.

but at $k'a = n\pi$, $T(E) \rightarrow 1$, we get perfect transmission.



$$E = \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\hbar^2 k'^2}{2m} - V_0$$

$$S(E) \approx \frac{e^{-ika}}{\cos k'a} \frac{1}{1 - \frac{i}{2} \left(\frac{k'}{k} + \frac{k}{k'} \right) \tan k'a},$$

Next, we expand around $k'a = n\pi$

$$\left(\frac{k}{k'} + \frac{k'}{k} \right) \tan k'a \approx \frac{4}{\Gamma_n} (E - E_n)$$

$$\frac{4}{\Gamma_n} = \frac{d}{dE} \left[\left(\frac{k}{k'} + \frac{k'}{k} \right) \tan k'a \right] \Big|_{k'=k_n}$$

$$\left\{ \begin{aligned} E_n &= \frac{\hbar^2 k_n^2}{2m} \end{aligned} \right.$$

$$e^{-ika} \sim 1, \quad \cos k'a \approx \pm 1$$

$$\frac{4}{\Gamma_n} = \left(\frac{k}{k'} + \frac{k'}{k} \right) \frac{d}{dE} \tan k'a \Big|_{E=E_n} = \left(\frac{k}{k'} + \frac{k'}{k} \right) \left(\sec^2 k'a \frac{dk'a}{dE} \right) \Big|_{E=E_n}$$

$$(\because \tan k'_n a = 0)$$

$$= \left(\frac{k}{k'} + \frac{k'}{k} \right) \frac{dk'a}{dE} \Big|_{E=E_n}$$

AMPAD

consider the case $k' \gg k$ we have $\frac{4}{\Gamma_n} \approx \frac{ak'}{k} \frac{dk'}{dE} \Big|_{E=E_n} \approx \frac{a}{k} \frac{m}{\hbar^2}$

$$\approx \frac{ma}{\hbar^2} \sqrt{\frac{\hbar^2}{2mE_n}} = \frac{a}{\hbar} \sqrt{\frac{m}{2E_n}}$$

$$\Rightarrow \Gamma_n \approx \frac{4\hbar}{a} \sqrt{\frac{2E_n}{m}} = (E_n \frac{\hbar^2}{2ma^2})^{1/2} \cdot 8$$

$$\approx 8(E_n \cdot E_k)^{1/2} \quad \text{where } E_k = \frac{\hbar^2}{2ma^2}. \quad \text{we consider } V_0 \gg E_n \gg E_k$$

Then $S(E) \approx \frac{e^{ika}}{\cos k'a} \frac{1}{1 - \frac{i2}{\Gamma_n}(E-E_n)} \approx \pm \frac{i\Gamma_n/2}{(E-E_n) + i\Gamma_n/2}$

and $T(E) \approx \frac{(\Gamma_n/2)^2}{(E-E_n)^2 + (\Gamma_n/2)^2}$ Breit - Wigner - formula

$S(E)$ has a pole at $E = E_n - i\Gamma_n/2$, however, this pole is not on the physical sheet of E .

$$S = e^{ika} \frac{1}{\cos \sqrt{\frac{2m(E+|V_0|)}{\hbar^2}} a - \frac{i}{2} \left(\sqrt{\frac{E}{E+|V_0|}} + \sqrt{\frac{E+|V_0|}{E}} \right) \sin \sqrt{\frac{2m(E+|V_0|)}{\hbar^2}} a}$$

The pole is reached if we interperate $\sqrt{E} \approx \sqrt{E_n} (1 - \frac{i\Gamma_n}{2E_n})$, thus $\sqrt{E}$ is defined on the second Riemann sheet!

①

Gamow - boundary condition

We start with $\psi(x) = \begin{cases} e^{ikx} + R(E) e^{-ikx} & x \rightarrow -\infty \\ S(E) e^{ikx} & x \rightarrow +\infty \end{cases}$

We assume there exist a pole

$$S(E) \sim C_S / E - E', \quad R(E) \sim C_R / E - E'$$

where $E' = \text{Re } E' - iP/2$, and E' lies on the 2nd Riemann sheet.

$$\psi/S \rightarrow \begin{cases} 1/S e^{ikx} + R/S e^{-ikx}, & x \rightarrow -\infty \\ e^{ikx}, & x \rightarrow +\infty \end{cases}$$

as $E \rightarrow E'$, the incoming wave disappears

$$\psi_{\text{pole}}(x) \sim \begin{cases} C_L e^{-ikx} & x \rightarrow -\infty \\ C_R e^{ikx} & x \rightarrow +\infty \end{cases} \quad \begin{array}{l} \text{outgoing} \\ \text{boundary} \end{array}$$

S-matrix: pole $\iff$ non-zero solution with
no-incoming wave

If E is real, ~~then the probability of particles has to be conserved~~, then $\psi(x,t) = \psi(x) e^{-iEt/\hbar}$, the probability in the central region ~~is~~ is conserved. But $j_R > 0$, $j_L < 0$, which means that probability decays.

Hence, E has to be complex. $e^{-iEt/\hbar} = e^{-iE't/\hbar - P't/2\hbar}$

We set E on the 2nd Riemann sheet

(2)

$$\Rightarrow k' = k - i\kappa \quad \text{with } \kappa > 0$$

the outgoing wave on the right side $e^{ik'x} = e^{ikx + \kappa x}$

which increases as x increases, i.e. $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Similar analysis applies as $x \rightarrow -\infty$!

Such a state does not belong to Hilbert space of L^2

How to interpret?

$$\psi(x, t) = e^{i(k'x - E't/\hbar)} = e^{i(kx - Et/\hbar)} e^{\kappa x - \frac{P}{2\hbar}t}$$

$$k'^2 = k^2 - \kappa^2 - 2i\kappa k$$

$$\text{Im} E = -\frac{\hbar^2}{m} \kappa k = -\frac{P}{2} \Rightarrow \frac{P}{2\hbar} = \frac{\hbar k \kappa}{m} = v\kappa$$

$$\psi(x, t) = e^{i(kx - Et)/\hbar} e^{\kappa(x - vt)}$$

If you find a particle at x , it should have come from $x=0$.

Hence, the time should be $t = x/v$, such that the time decay is compensated by the spatial increase.

⊗ Resonance revisits - outgoing eigenmode

Outgoing in both directions

$$\psi(x) = \begin{cases} C_L e^{-ikx} & x < -a/2 \\ \psi_{in}(x) & |x| < a/2 \\ C_R e^{ikx} & x > a/2 \end{cases}$$

① even parity $\psi_{in}(x) = A \cos qx$

$$\text{match boundary condition} \Rightarrow \psi'/\psi = \left. \frac{-q \sin qx}{\cos qx} \right|_{x=a/2} = ik$$

$$\boxed{-q \tan \frac{q}{2} a = ik}$$

For energy $E \ll |V_0|$, i.e. $k/q \ll 1$. we have $\frac{qa}{2} \simeq n\pi$,

$$\Rightarrow qa \simeq 2n\pi. \text{ Hence } \psi_{in}(x) \simeq \sqrt{\frac{2}{a}} \cos \frac{2n\pi}{a} x.$$

calculation of the imaginary part

$$\frac{q}{2} a = n\pi + \delta \quad \text{where } |\delta| \ll 1$$

$$\tan \frac{q}{2} a = \tan \delta = \frac{ik}{-q} \Rightarrow \delta \simeq -ik/q$$

$$\Rightarrow qa \simeq 2n\pi - ik/q_0, \quad q_0 = \frac{2n\pi}{a}$$

~~$$\Gamma = \frac{4i}{2n\pi q_0^2} - 2i\gamma > 0 \quad \frac{4i^2}{2m} = \frac{4n^2}{2m} - i\frac{4k}{2} \Rightarrow \Gamma = \frac{4n^2}{2m} - i\frac{4k}{2}$$~~

$$q \simeq q_0 - i2k/q_0 a \quad q^2 \simeq q_0^2 - 4i k/a$$

$$\Rightarrow E_n = \frac{\hbar^2}{2m} q^2 - V_0 \Rightarrow \text{Im} E = \frac{\hbar^2}{2m} (-4i k/a) = -\frac{i\Gamma_n}{2}$$

$$= \text{Re} E - i\Gamma_n/2 \quad \Rightarrow \quad \Gamma_n = \frac{4\hbar^2 k}{ma} \quad \left. \begin{array}{l} \\ E_n = \frac{\hbar^2 k^2}{2m} \end{array} \right\} \Rightarrow \Gamma_n = \frac{4\hbar}{a} \sqrt{\frac{2E_n}{m}}$$

This is just the resonance width of Breit-Wigner.

and the pole is in the 2nd Riemann sheet

Consider time-evolution $x \gg a/2$

$$\psi_R(x,t) \sim \int dE \frac{e^{i k(E)(x-a/2) - iEt/\hbar}}{E - E_0 + i\Gamma/2}$$

$$k(E) \simeq k_0 + (E - E_0)/\hbar v$$

$$k(E)(x-a/2) - Et/\hbar = (k_0(x-a/2) - E_0 t/\hbar) + (E - E_0) \left(\frac{y}{t-y/v} \right)$$

$y = x - a/2$

$$\Rightarrow \psi_R(x,t) \sim e^{i(k_0 y - E_0 t/\hbar)} \int dE \frac{e^{-i(E-E_0)(t-y/v)}}{E - E_0 + i\Gamma/2}$$

$$\sim e^{i(k_0 y - E_0 t/\hbar)}$$

$$\cdot \exp \left[-\left(\frac{\Gamma}{2} (t - y/v) \right) \right]$$

