

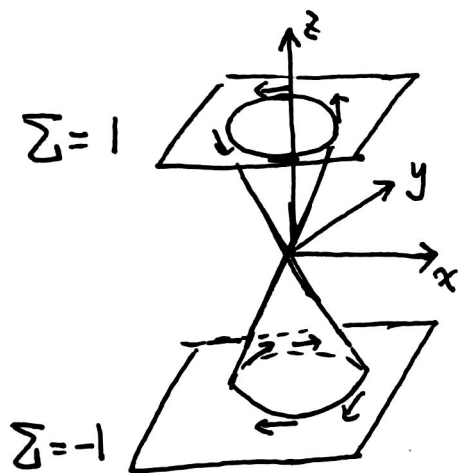
Lecture 9 More on Landau levels

{ Laughlin argument for integer QHE

{ 1+1D chiral anomaly

{ Landau levels in 3D

$$H_{3D} = \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2 z^2 - \omega z (p_x \sigma_y - p_y \sigma_x)$$



spatial separation of helical mode

$$\Sigma = \hat{p}_x \sigma_y - \hat{p}_y \sigma_x$$

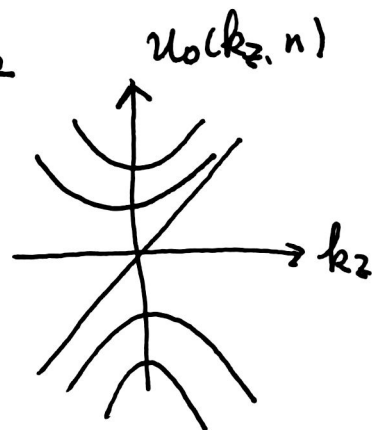
{ Landau level in 4D and 4D QHE

$$H_{4D} = \frac{\vec{p}_u^2}{2m} + \frac{1}{2}m\omega^2 u^2 + \frac{\vec{p}_{3D}^2}{2m} - \omega u \vec{p}_{3D} \cdot \vec{\sigma}$$

$$\rightarrow \frac{\vec{p}_u^2}{2m} + \frac{1}{2}m\omega_{s0}^2 \left(u - \frac{l_{s0}^2}{\hbar} (-i\hbar \vec{D} \cdot \vec{\sigma}) \right)^2$$

$$-i\hbar \vec{D} \cdot \vec{\sigma} = -i\hbar \left[\partial_i - \frac{ie}{\hbar c} A_i \right] \sigma_i$$

$$\Rightarrow J_u = \alpha \frac{e}{4\pi^2 \hbar} \mathbf{E} \cdot \mathbf{B}$$

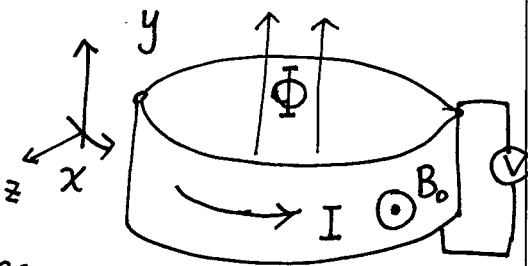


04/04/2012

①

Laughlin argument for integer Quantum Hall

Goal: relate the current I carried around the loop (cylindrical geometry) to the voltage drop across the two edges in the y -direction.



$$\vec{A} = -B_0 y \hat{e}_x$$

Now let us thread a new flux Φ in the center of the cylinder.

$$\begin{aligned} \Rightarrow \vec{B} &= \partial_x A_y - \partial_y A_x \\ &= B_0 \hat{e}_z \end{aligned}$$

The corresponding $\vec{A}' = \frac{\Phi}{L_x} \hat{e}_x$, where L_x is the circumference around $\hat{e}_x$ -direction.

The current along the x -direction

$$I = c \frac{\partial E}{\partial \Phi} = \frac{c}{L_x} \frac{\partial E}{\partial A}$$

c.f. $I = j_x \cdot L_y \Rightarrow \frac{1}{c} j_x \cdot A (L_x L_y) = \Delta E$

For non-interaction electrons

$$H = \frac{1}{2m} \left(p_x - \frac{e}{c} A_x \right)^2 + \frac{1}{2m} p_y^2 + e E_0 y \quad \text{where } E_0 = \frac{\Delta V}{L_y}$$

$$\psi_{n, k_x}(\vec{r}) = e^{ik_x x} \phi_n(y - y_0(k_x))$$

$$\left[\frac{1}{2m} \left(\hbar k_x - \frac{eB}{c} y \right)^2 + \frac{p_y^2}{2m} + e E_0 y \right] \phi_n(y - y_0(k_x)) = E_n \phi_n(y - y_0(k_x))$$

$$\frac{m}{2} \omega^2 \left[y - \frac{\hbar c}{eB} k_x \right]^2 + eE_0 y = \frac{m}{2} \omega^2 \left[y - \frac{\hbar c}{eB} k_x + \frac{eE_0}{m\omega^2} \right]^2$$

$$\boxed{\omega = \frac{eB}{mc}}$$

$$+ \frac{\hbar c}{eB} k_x eE_0 - \frac{1}{2} \frac{e^2 E_0^2}{m\omega^2}$$

$$= \frac{m}{2} \omega^2 \left[y - y_0(k_x) \right]^2 + eE_0 \left[\underbrace{\frac{\hbar c}{eB} k_x - \frac{eE_0}{m\omega^2}}_{y_0(k_x)} \right] + \frac{1}{2} \frac{eE_0}{m\omega^2}$$

$$= \frac{m}{2} \omega^2 \left[y - y_0(k_x) \right]^2 + eE_0 y_0(k_x) + \frac{mc^2}{2} \left[\frac{E_0}{B} \right]^2$$

$$y_0(k_x) = \frac{\hbar c}{eB} k_x - \frac{eE_0}{m\omega^2} = \frac{1}{eB/mc} \left[\frac{\hbar k_x}{m} - \frac{E_0 c}{B_0} \right] = \frac{1}{\omega} \left[\frac{\hbar k_x}{m} - c \frac{E_0}{B_0} \right]$$

$$\Rightarrow E_n = (n + 1/2) \hbar \omega + eE_0 y_0(k_x) + \frac{1}{2} mc^2 \left(\frac{E_0}{B_0} \right)^2$$

Let us thread the extra flux $\Delta A \hat{e}_x = \frac{\Phi}{L_x} \hat{e}_x \Rightarrow A = B_0 \left(y + \frac{\Delta A}{B_0} \right)$

$$\Rightarrow \frac{m}{2} \omega^2 \left[y - \frac{\hbar c}{eB} k_x + \frac{\Delta A}{B_0} \right]^2 + eE_0 y$$

$$= \frac{m}{2} \omega^2 \left[y - \frac{\hbar c}{eB} k_x + \frac{eE_0}{m\omega^2} + \frac{\Delta A}{B_0} \right]^2 + eE_0 \left[\frac{\hbar c}{eB} k_x - \frac{\Delta A}{B_0} \right] - \frac{eE_0}{m\omega^2}$$

$$= \frac{m}{2} \omega^2 \left[y - y'_0(k_x) \right]^2 + eE_0 y'_0(k_x) + \frac{m\omega^2}{2} \left[\frac{eE_0}{m\omega^2} \right]^2 + \frac{m}{2} c^2 \left[\frac{E_0}{B_0} \right]^2$$

with $y'_0(k_x) = y_0(k_x) - \frac{\Delta A}{B_0}$

Let change $\Delta A \cdot L_x = \Delta \phi = \frac{hc}{e}$, the system comes back to itself. The energy change is due to pump a particle between two edges

$$\Delta E = eV \Rightarrow I = C \frac{eV}{hc/e} = \frac{e^2}{h} V \Rightarrow \sigma_H = \frac{e^2}{h}$$

If there are several sets of Landau level $\Rightarrow \sigma_H = \frac{ne^2}{h}$.

★ For dirty and interacting systems

As long as, the Fermi surface lies inside the gap, thread a

$\Delta \phi = \frac{hc}{e}$ adiabatically cannot generate a particle-hole excitation

across the gap, thus the bulk should remain the same as ~~as~~ before flux threading. The only possible place to change is the

edge, which is gapless. It removes a particle from one edge, moving it into the bulk, and extra another one from the bulk to another edge. $\rightarrow$ as a pump.

Chiral anomaly 1+1 D

$$H_R = v \cdot (p - \frac{e}{c} A(x)) \quad , \quad H_L = -v \cdot p = -v(p - \frac{e}{c} A(x))$$

right mover

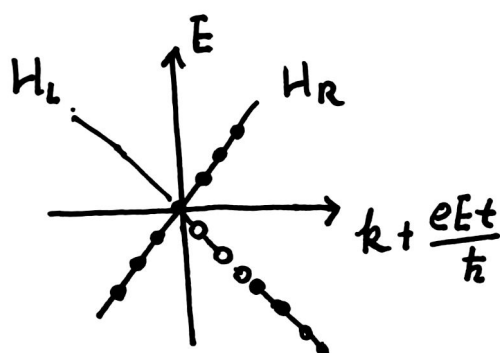
left movers.

fermion : $A = -cEt \Rightarrow E = -\frac{1}{c} \frac{\partial A}{\partial t}$

In the adiabatic limit, i.e. $E \rightarrow 0$, the process is a

slow pumping:

$$\Delta N_R(t) = \Delta k \cdot \frac{2\pi}{L}$$



$$d\Delta k/dt = \frac{eE}{\hbar}$$

$$\Rightarrow dN_R(t)/dt = \frac{eE}{\hbar} \cdot \frac{L}{2\pi}$$

$$dN_L(t)/dt = -\frac{eE}{\hbar} \cdot \frac{L}{2\pi}$$

Hence, left and right movers' numbers are not separately conserved. Define $j_5 = \dots (N_R - N_L)/L$

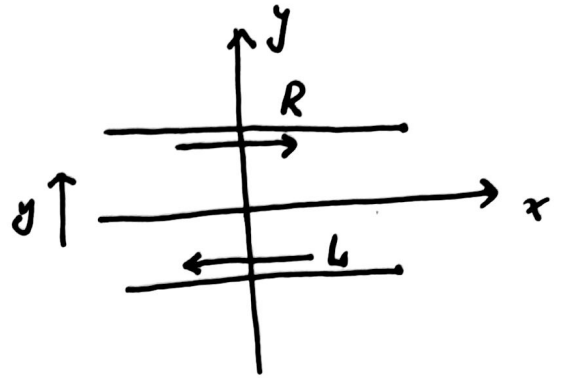
which is just electric current/e

$$\boxed{\frac{dj_5}{dt} = \frac{eE_x}{\pi\hbar}}$$

connect to QH response

$$j_y = \frac{e \Delta N_R}{L \Delta t} = \frac{-e \Delta N_L}{L \Delta t}$$

$$= \frac{e^2 E_x}{2\pi\hbar} \Rightarrow \boxed{\sigma_{xy} = \frac{e^2}{h}}$$



• Digression on 1+1D Schwinger model.

continuity equation: $\frac{\partial}{\partial t} \Delta j_0 + \frac{\partial}{\partial x} (v j_5) = 0$ ①

where $j_0 = (N_R + N_L)/L$ $j_5 = (N_R - N_L)/L$

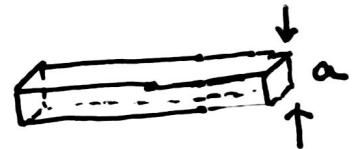
$v \frac{\partial}{\partial x} (j_R) + (-v) \frac{\partial}{\partial x} (-j_L)$

Electric field $\frac{d}{dt} j_5 = \frac{\partial}{\partial t} j_5 + v \frac{\partial}{\partial x} (\Delta j_0) = \frac{e E_x}{\pi\hbar}$ ②

1+1D Maxwell equation

③ $\partial_x E_x = 4\pi e \Delta j_0 / a^2$

$\frac{1}{c} \partial_t E_x = -\frac{4\pi e j_5 v}{c a^2} \leftarrow \nabla \times B = 4\pi e j_5 + \frac{1}{c} \frac{\partial}{\partial t} E$



④ $\Rightarrow \partial_t E_x = -4\pi e v j_5 / a^2$

There are three of them independent.

$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_x = -\frac{4\pi e}{c} \frac{v}{a^2} \quad \frac{1}{c} \frac{\partial}{\partial t} j_5 = -\frac{4\pi e}{c^2} \frac{v}{a^2} \left[\frac{e E_x}{\pi\hbar} - v \frac{\partial}{\partial x} (\Delta j_0) \right]$

$$\partial_x^2 E_x = \frac{4\pi e}{a^2} (\partial_x \Delta j_0) \Rightarrow \partial_x (\Delta j_0) = \frac{a^2}{4\pi e} \partial_x^2 E_x$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_x - \frac{4\pi e}{c^2 a^2} v^2 \frac{a^2}{4\pi e} \partial_x^2 E_x = - \frac{4\pi e^2 v}{a^2 c^2} \frac{E_x}{\pi \hbar}$$

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) E_x = - \frac{4\pi}{a^2} \frac{e^2}{\pi \hbar v} E_x$$

$$\left\{ \left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) + \frac{4}{a^2} \left(\frac{c}{v} \right) \alpha \right\} E_x = 0, \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$$

$$\Rightarrow - \frac{\omega^2}{v^2} + k^2 + m^2 = 0 \Rightarrow$$

$$\omega^2 = k^2 v^2 + m^2 v^2 \quad \text{with}$$

$$\boxed{\omega^2 = k^2 v^2 + \frac{4}{a^2} \alpha c v}$$

* Landau level in 3D

Let us consider how to generalize LL from 2D to 3D.

In 2D, $\rightarrow$ spatial separation of $\underbrace{\text{chiral modes}}_{1D \text{ x}}$ along y -direction.

But in 3D, we can set (k_x, k_y) as plane-wave mode.

Then we have to decide which one to move to $+\hat{z}$, and

which one to move to $-\hat{z}$. However, we have (k_x, k_y) ,

and here is the trouble. For this purpose, we

introduce 2D Dirac equation

$$H_{2D} = p_x \sigma_y - p_y \sigma_x = \begin{pmatrix} 0 & -p_y - i p_x \\ -p_y + i p_x & 0 \end{pmatrix}$$

Try solution $\psi_{p_x, p_y, \pm}(x) = e^{i \vec{p}_\parallel \cdot \vec{x}} \chi_\pm$

$$p_x = p \cos \phi_p, \quad p_y = p \sin \phi_p \quad \Rightarrow \quad \begin{aligned} -p_y - i p_x &= p(-\sin \phi - i \cos \phi) \\ &= -i p e^{i \phi_p} \end{aligned}$$

$$\begin{aligned} -p_y + i p_x &= i(p_x + i p_y) \\ &= i p e^{i \phi_p} \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 0 & -i e^{i \phi_p} \\ i e^{i \phi_p} & 0 \end{pmatrix} \chi_\pm(\vec{p}) = \pm \chi_\pm(\vec{p})$$

$$\chi_\pm(\vec{p}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i \phi/2} \\ \pm i e^{i \phi/2} \end{pmatrix} \Rightarrow \psi_{p_x, p_y, \pm}(x, y) = e^{i \vec{p}_\parallel \cdot \vec{x}} \chi_\pm(p_\parallel)$$

$$\boxed{H_{2D} \psi_{p_x, p_y, \pm} = \pm p_\parallel \psi_{p_x, p_y}(x, y)}$$

Here, $\pm$ refer to positive helicity and negative one. ⁸

We can shift upwards and downwards according to Σ 's

eigenvalue, $\hat{\Sigma} = \frac{1}{P} \begin{pmatrix} 0 & -P_y - iP_x \\ -P_y + iP_x & 0 \end{pmatrix}$.

Then we define 3D LL

$$H_{3D} = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + \frac{1}{2} m \omega^2 z^2 - \omega z (P_x \sigma_y - P_y \sigma_x)$$

Try solution $\psi_{n, \vec{p}, \pm}(\vec{r}) = e^{i\vec{p} \cdot \vec{r}_0} \cdot \phi_n(z) \otimes \chi_{\pm}(\vec{p})$

$$\Rightarrow H_{3D} = \frac{P_z^2}{2m} + \frac{1}{2} m \omega^2 \left[z - \frac{1}{m\omega} (P_x \sigma_y - P_y \sigma_x) \right]^2$$

$$H_{3D} \psi_{n, \vec{p}, \pm}(\vec{r}) = \left\{ \frac{P_z^2}{2m} + \frac{1}{2} m \omega^2 \left[z - \frac{\hbar}{m\omega} (k)(\pm) \right] \right\} \psi_{n, \vec{p}, \pm}(\vec{r})$$

$$= E_n \psi_{n, \vec{p}, \pm}(\vec{r})$$

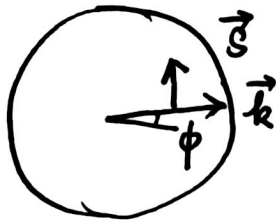
$$\left\{ \frac{P_z^2}{2m} + \frac{1}{2} m \omega^2 \left[z - \frac{\hbar}{m\omega} k(\pm) \right]^2 \right\} \phi_n(z) = E_n \phi_n(z)$$

$$\Rightarrow \boxed{z_0(k) = \pm k \ell^2} \quad \text{helicity replaces the role of chirality}$$

$$E_n = (n + 1/2) \hbar \omega \quad \text{independent of } \vec{p} \text{ and } \Sigma = \pm.$$

● what kind of states are pushed up?

$$\Sigma = 1$$



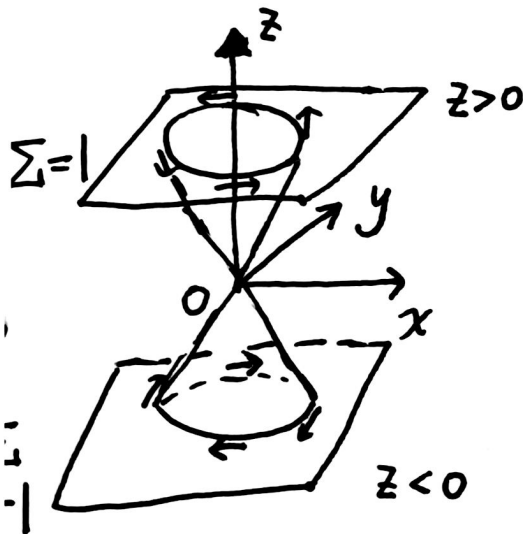
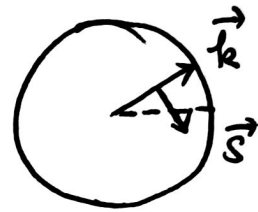
$$\chi_+^\dagger \sigma_x \chi_+ = \frac{1}{2} (e^{i\phi/2}, -ie^{-i\phi/2}) (1 \ 1) \begin{pmatrix} e^{-i\phi/2} \\ ie^{i\phi/2} \end{pmatrix}$$

$$= \frac{1}{2} [e^{i\phi} - i e^{-i\phi}] = -\sin \phi$$

$$\chi_+^\dagger \sigma_y \chi_+ = \frac{1}{2} (e^{i\phi/2}, -ie^{-i\phi/2}) (i \ -i) \begin{pmatrix} e^{-i\phi/2} \\ ie^{i\phi/2} \end{pmatrix}$$

$$= \frac{1}{2} (e^{i\phi} + e^{-i\phi}) = \cos \phi$$

● $\Sigma = -1 \Rightarrow \begin{cases} \chi_-^\dagger \sigma_x \chi_- = \sin \phi \\ \chi_-^\dagger \sigma_y \chi_- = -\cos \phi \end{cases}$



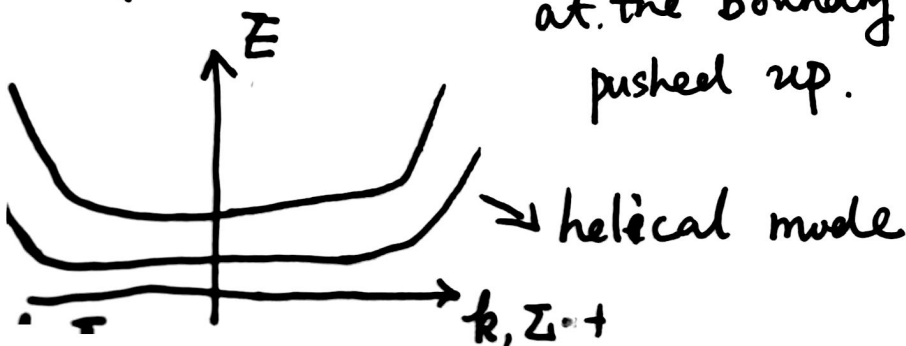
spatial separation
of helical mode

If we impose open boundary
condition along z , i.e.,

$$V(z) = \begin{cases} +\infty, & |z| > \frac{L}{2} \\ 0, & |z| < L/2 \end{cases}$$

Then for $k \cdot L^2 > L/2$, their
states with

central position $z_{0,\pm}(k)$ is pinned
at the boundary, such that $E_{n,\pm}(k)$ is
pushed up.



(*) Landau level in 4D

Similarly, we can define LL in 4D by the same analogy.

We introduce the 3D Weyl Equation

$$H_{3D} = \vec{p} \cdot \vec{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = p \begin{pmatrix} \hat{p}_3 & \hat{p}_1 - i\hat{p}_2 \\ \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 \end{pmatrix}$$

Define projection operator $P_{\pm} = \frac{1 \pm \hat{p} \cdot \vec{\sigma}}{2}$

$$P_{+} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi_{+}(\hat{p}) \Rightarrow \chi_{+}(\hat{p}) = N \begin{pmatrix} 1 + \hat{p}_3 & \hat{p}_1 - i\hat{p}_2 \\ \hat{p}_1 + i\hat{p}_2 & 1 - \hat{p}_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

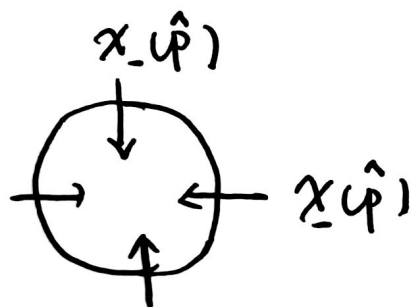
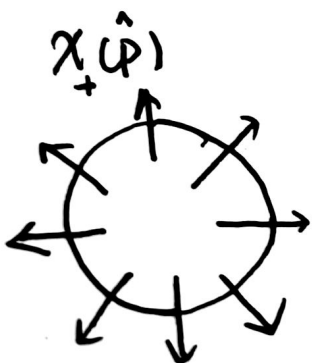
$$= N \begin{pmatrix} 1 + \hat{p}_3 \\ \hat{p}_1 + i\hat{p}_2 \end{pmatrix}$$

$$N^2 [1^2 + \hat{p}_3^2 + 2\hat{p}_3 + \hat{p}_1^2 + \hat{p}_2^2] = 1 \Rightarrow N = \frac{1}{\sqrt{2(1 + \hat{p}_3)}}$$

$$\Rightarrow \chi_{+}(\hat{p}) = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\varphi} \end{pmatrix}, \text{ satisfying } (\hat{p} \cdot \vec{\sigma}) \chi_{+}(\hat{p}) = \chi_{+}(\hat{p})$$

similarly, we define $P_{-} = \frac{1 - \hat{p} \cdot \vec{\sigma}}{2} \Rightarrow \chi_{-}(\hat{p}) = N \begin{pmatrix} \hat{p}_1 - i\hat{p}_2 \\ 1 - \hat{p}_3 \end{pmatrix}$

$$\Rightarrow \chi_{-}(\hat{p}) = \begin{pmatrix} -\sin \frac{\theta}{2} e^{i\varphi} \\ \cos \frac{\theta}{2} \end{pmatrix} \text{ with } (\hat{p} \cdot \vec{\sigma}) \chi_{-}(\hat{p}) = -\chi_{-}(\hat{p})$$



$$\hat{\Sigma} = \hat{p} \cdot \vec{\sigma}$$

$$\Rightarrow \Sigma = \pm 1$$

Then we have the 4D LL

$$H_{LL}^{4D} = \frac{P_u^2}{2m} + \frac{1}{2} m \omega^2 u^2 + \frac{\vec{P}_{3D}^2}{2m} - \omega u \vec{P}_{3D} \cdot \vec{\sigma}$$

$$= \frac{P_u^2}{2m} + \frac{1}{2} m \omega^2 \left[u - \ell^2 P_{3D}/\hbar \cdot \hat{P} \cdot \vec{\sigma} \right]^2 \quad \ell = \sqrt{\frac{\hbar}{m\omega}}$$

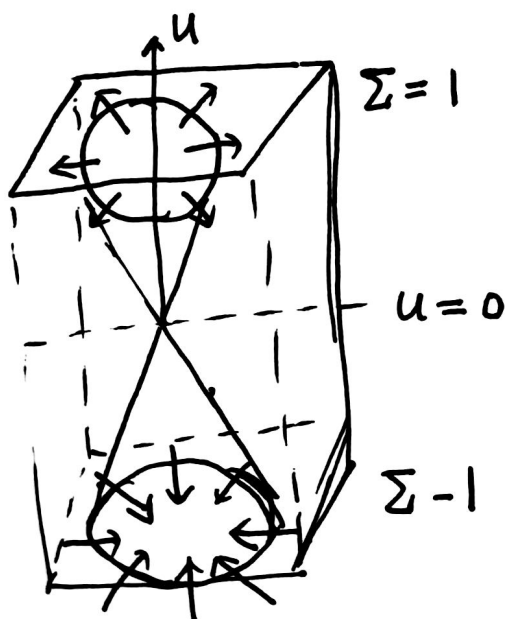
Similarly, we try $\psi_{n, \vec{P}_{3D}, \pm}(\vec{r}, u) = e^{i \vec{P}_{3D} \cdot \vec{r}} \phi_n(u) \otimes \chi_{\pm}(\hat{P})$

$$H_{4D} \psi_{n, \vec{P}_{3D}, \pm} = \left(\frac{P_u^2}{2m} + \frac{1}{2} m \omega^2 \left[u - \frac{\ell^2 P_{3D}}{\hbar} (\pm) \right]^2 \right) \psi_{n, \vec{P}_{3D}, \pm}$$

$$= E_n \psi_{n, \vec{P}_{3D}, \pm} \quad \leftarrow (n + 1/2) \hbar \omega$$

$$\Rightarrow \left(\frac{P_u^2}{2m} + \frac{1}{2} m \omega^2 [u - u_0]^2 \right) \phi_n(u) = E_n \phi_n(u)$$

$$u_{0, \pm}(p) = \pm P \ell^2 / \hbar$$



spatial separation
of Weyl modes with
opposite helicities!

4D Landau level with B-field along the z-axis

$$H = \frac{p_u^2}{2m} + \frac{1}{2} m \omega_{so}^2 \left[u - \frac{l_{so}^2}{\hbar} (-i\hbar \vec{D} \cdot \vec{\sigma}) \right]^2$$

where $l_{so} = \sqrt{\frac{\hbar}{m\omega_{so}}}$

we first look at eigenvalues of $-i\hbar \vec{D} \cdot \vec{\sigma} = -i\hbar \left[\partial_i - \frac{ie}{\hbar c} A_i \right] \sigma_i$

$$A_x = 0, A_y = Bx$$

$$\begin{bmatrix} -i\hbar \partial_z & -i\hbar \partial_x - \hbar \partial_y + i \frac{e}{c} Bx \\ -i\hbar \partial_x + \hbar \partial_y - i \frac{e}{c} Bx & -i\hbar \partial_z \end{bmatrix} \chi_n = \lambda_n \chi_n$$

$$\chi_n = e^{ik_z z + ik_y y} \chi_n(x), \quad \text{define } l_B^2 = \frac{\hbar c}{|eB|}$$

$$\begin{bmatrix} \hbar k_z & -i\hbar \partial_x - i\hbar k_y + i\hbar l_B^{-2} x \\ -i\hbar \partial_x + i\hbar k_y & -i\hbar l_B^{-2} x - \hbar k_z \end{bmatrix} \chi_n(x) = \lambda_n \chi_n(x)$$

define $a^\dagger = \frac{1}{\sqrt{2} l_B} \left[(x - l_B^2 k_y) - l_B^2 \partial_x \right]$

$$a = \frac{1}{\sqrt{2} l_B} \left[(x - l_B^2 k_y) + l_B^2 \partial_x \right]$$

$$\begin{bmatrix} \hbar k_z & \frac{i\hbar\sqrt{2}}{l_B} a^\dagger \\ -\frac{i\hbar\sqrt{2}}{l_B} a & -\hbar k_z \end{bmatrix} \chi_n(x-x_0) = \lambda_n \chi_n(x-x_0)$$

where $\underline{x_0(k_y)} = \underline{l_B^2 k_y}$.

express $\chi_n(x-x_0) = \begin{bmatrix} \alpha_n \phi_n(x-x_0) \\ \beta_n \phi_{n-1}(x-x_0) \end{bmatrix}$

$$\Rightarrow \hbar k_z \phi_n(x-x_0) \alpha_n + \frac{i\hbar}{l_B} \sqrt{2} \sqrt{n} \phi_n(x-x_0) \beta_n = \lambda_n \alpha_n \phi_n$$

$$-\frac{i\hbar}{l_B} \sqrt{2n} \underbrace{\phi_n}_{\alpha_n} - \hbar k_z \underbrace{\phi_{n-1}}_{\beta_n} = \lambda_n \beta_n \phi_{n-1}$$

$$\Rightarrow (\hbar k_z - \lambda_n) \alpha_n + \frac{i\hbar}{l_B} \sqrt{2n} \beta_n = 0$$

$$-\frac{i\hbar}{l_B} \sqrt{2n} \alpha_n - (\hbar k_z + \lambda_n) \beta_n = 0$$

$$\Rightarrow \begin{vmatrix} \hbar k_z - \lambda_n & \frac{i\hbar}{l_B} \sqrt{2n} \\ -\frac{i\hbar}{l_B} \sqrt{2n} & -(\hbar k_z + \lambda_n) \end{vmatrix} = 0 \Rightarrow -(\hbar k_z)^2 + \lambda_n^2 - \left(\frac{\hbar}{l_B}\right)^2 \cdot 2n = 0$$

$$\lambda_n = \pm \hbar \sqrt{k_z^2 + 2n/l_B^2}$$

for $n=0$ channel $\Rightarrow \begin{bmatrix} \hbar k_z & \frac{i\hbar\sqrt{2}}{l_B} a^\dagger \\ -\frac{i\hbar\sqrt{2}}{l_B} a & -\hbar k_z \end{bmatrix} \begin{bmatrix} \phi_0(x-x_0) \\ 0 \end{bmatrix} = \lambda_0 \begin{bmatrix} \phi_0(x-x_0) \\ 0 \end{bmatrix}$

$$\Rightarrow \lambda_0 = \hbar k_z \quad \text{only one branch}$$

$$\text{from } -\frac{i\hbar}{l_B} \sqrt{2n} \alpha_n - (\hbar k_z + \lambda_n) \beta_n = 0$$

$$\Rightarrow \frac{\beta_n}{\alpha_n} = \frac{-i\hbar/l_B \sqrt{2n}}{\hbar k_z \pm \hbar \sqrt{k_z^2 + 2n/l_B^2}} = \frac{-i\sqrt{2n}}{k_z \pm \sqrt{k_z^2 + 2n/l_B^2}}$$

Summary:

$$\text{For } n=0, \lambda_0 = \hbar k_z, \chi_0(x-x_0) = \begin{bmatrix} \phi_0(x-x_0) \\ 0 \end{bmatrix}$$

$$n \neq 0, \lambda_{\pm n} = \pm \hbar \sqrt{k_z^2 + \frac{2n}{l_B^2}}, \chi_{\pm n} = \begin{bmatrix} \alpha_{\pm n} \phi_n(x-x_0) \\ \beta_{\pm n} \phi_{n-1}(x-x_0) \end{bmatrix}$$

$$\text{with } \frac{\beta_{\pm n}}{\alpha_{\pm n}} = \frac{-i\sqrt{2n}}{k_z \pm \sqrt{k_z^2 + 2n/l_B^2}}$$

write the general solution to the full Eq

$$\psi(k_y, k_z) = e^{ik_y y + ik_z z} \phi_m[u - u_0(k_z, n)] \otimes \chi_n(x-x_0)$$

$$(-i\hbar \vec{D} \cdot \vec{\sigma}) e^{ik_y y + ik_z z} \chi_n(x-x_0) = u_0(k_z, n) \chi_n(x-x_0)$$

$$u_0(k_z) = l_B^2 k_z$$

$$\text{and } u_0(k_z, n) = \begin{cases} k_z l_B^2 \\ \pm \sqrt{k_z^2 + 2n/l_B^2} l_B^2 \end{cases}$$

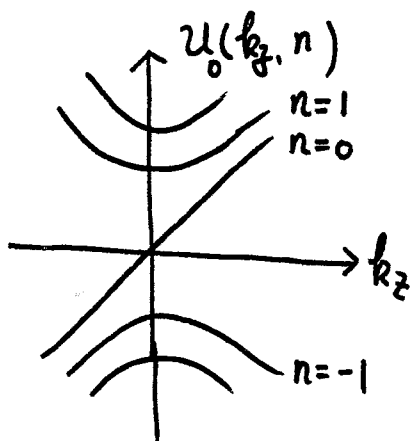
we have

$$\left\{ \frac{p_u^2}{2m} + \frac{1}{2} m \omega_{so}^2 [u - u_0(k_z, n)] \right\} \phi_m(u) = E_m \phi_m(u)$$

$\Rightarrow E_m = (m + 1/2) \hbar \omega_{so}$, which is not affected by the magnetic field.

$$\phi_m(u) = \phi_m(u - u_0(k_z, n))$$

$\uparrow$
harmonic oscillator WF with center at $u_0(k_z, n)$



$$u_0(k_z, 0) = k_z^2 l_B^2$$

$$u_0(k_z, n) = \pm \sqrt{k_z^2 + \frac{2n}{l_B^2}} l_B^2$$

Having understood these, we now add the electric field $\sqrt{E_0}$ along the u -axis. This time, we use the scalar potential

$$H^{4D} = \frac{p_u^2}{2m} + \frac{1}{2} m \omega_{so}^2 \left[u - \frac{l_{so}^2}{\hbar} (-i\hbar \vec{D} \cdot \vec{\sigma}) \right]^2 + e E_0 u$$

- The channel with $n=0$ (2D magnetic LL index) is special.

For each 4D LL index m , $\phi_u(u - u_0(k_z, 0))$

with $u_0(k_z, 0) = k_z l_B^2$.

Now turn on Electric field along z -axis, E_z , the acceleration

$$k_z(t) = k_z(0) + eE_z t / \hbar.$$

The central position $u_0(k_z, n=0)$ moves along the u -axis!

Consider a 3D cross section perpendicular to u -axis,

within the time interval Δt , the # of states of each filled

4D LL passing such a cross section

$$N = \frac{L_x L_y}{2\pi l_B^2} \cdot \frac{eE_z t}{\hbar} / \frac{2\pi}{L_z} = \frac{eE_z t}{l_B^2 \hbar (2\pi)^2} L_x L_y L_z$$

The current density

$$\dot{j}_u = \frac{e dN}{dt} / L_x L_y L_z = \frac{e^2 E_z}{4\pi^2 \hbar l_B^2} = \frac{e B_z E_z e^2}{4\pi^2 \hbar^2 c}$$

$$l_B^2 = \frac{\hbar c}{e B_z} = \frac{e^2}{\hbar c} \cdot \frac{e}{4\pi^2 \hbar} \vec{E} \cdot \vec{B}$$

$$\boxed{\dot{j}_u = \alpha \cdot \frac{e}{4\pi^2 \hbar} \vec{E} \cdot \vec{B}}$$