

Quantum Mechanics 2026 HW5

Due 10/19 in Class

October 13, 2026

Problem 1. Coherent states

Use the fact that $a|\alpha\rangle = \alpha|\alpha\rangle$. Prove that for the coherent state $|\alpha\rangle$, define $\overline{\Delta x^2} = \langle\alpha|x^2|\alpha\rangle - (\langle\alpha|x|\alpha\rangle)^2$, and $\overline{\Delta p^2} = \langle\alpha|p^2|\alpha\rangle - (\langle\alpha|p|\alpha\rangle)^2$. Prove that $\sqrt{\overline{\Delta x^2}}\sqrt{\overline{\Delta p^2}} = \hbar/2$. Explain why coherent states are called the most classical quantum state.

Problem 2. Wavefunctions of Harmonic Oscillator

In class we have derived that the ground state wavefunction of a harmonic oscillator satisfies

$$a\psi_0(x) = 0. \quad (1)$$

Start from here, please derive the normalized expression of $\psi_0(x)$. Then according to $|n\rangle = (a^\dagger)^n/\sqrt{n!}|0\rangle$, derive the normalized expression of the n -th excited wavefunction $\psi_n(x)$ for $n = 1, 2$.

Problem 3. High dimensional Oscillator

1) Consider the D -dimensional harmonic oscillators

$$H = -\sum_{i=1}^D \frac{\hbar^2 \nabla_i^2}{2m} + \frac{1}{2} m \omega^2 \sum_{i=1}^D x_i^2. \quad (2)$$

For each dimension, we define the creation and annihilation operators a_i and $a_i^\dagger$. Define $a = (a_1, a_2, \dots, a_D)^T$. Show that for an arbitrary special unitary transformation U satisfying $U^\dagger U = 1$ and $\det U = 1$, the Hamiltonian is invariant under the transformation $a' = Ua$.

2) This transformation is called the $SU(D)$ transformation. It can be viewed as a rotation in the D -dimensional complex space. If the rotation angle is small, we can write down $U = 1 - iA\theta$ where A is a $D \times D$ Hermitian matrix and $\text{tr} A = 0$. Prove that $Q_{ij} = a_i^\dagger A_{ij} a_j$ commutes with H .

2) For the case $D = 2$, we have found the three bases for A , which are the Pauli matrices. Write down the corresponding conserved quantities. Please find the bases for A for $D = 3$, and write down the corresponding conserved quantities.

Problem 4. Quantum Virial Theorem

1) Consider a particle in the 3D whose Hamiltonian is given by

$$H = p^2/2m + V(x). \quad (3)$$

By calculating $[\mathbf{x} \cdot \mathbf{p}, H]$, prove that

$$\frac{d}{dt} \langle \mathbf{x} \cdot \mathbf{p} \rangle = \langle p^2/m \rangle - \langle \mathbf{x} \cdot \nabla V \rangle. \quad (4)$$

To identify the relation with the classic Virial theorem, we need the left-hand-side vanish. Under what condition does it vanish?

2) Feynman-Hellman theorem: Consider a Hamiltonian $H(\lambda)$ with a parameter λ , whose normalized eigen-state $\psi_{n,\lambda}$ satisfies

$$H(\lambda)\psi_{n,\lambda} = E_n(\lambda)\psi_{n,\lambda}. \quad (5)$$

Prove that

$$\frac{\partial E_n(\lambda)}{\partial \lambda} = \langle \psi_{n\lambda} | \frac{\partial H}{\partial \lambda} | \psi_{n\lambda} \rangle. \quad (6)$$

3) Another way to prove Virial theorem is by Feynman-Hellman theorem. Define $H(\lambda)$ as the result of performing the a scaling transformation $x \rightarrow x' = \lambda x$ to $H(x)$. Write down the form of $H(\lambda)$. Then use the Feynman-Hellman theorem to prove the Virial theorem.

4) Apply the quantum Virial theorem to the harmonic oscillator. What is your conclusion?

Problem 5. Operator normal product

Consider $A = \alpha a + \alpha' a^\dagger$, and $B = \beta a + \beta' a^\dagger$. Define the normal product of e^A as

$$: e^A := e^{\alpha a + \alpha' a^\dagger} := e^{\alpha' a^\dagger} e^{\alpha a}. \quad (7)$$

Prove that $\langle 0 | : e^A : | 0 \rangle = 1$ where $|0\rangle$ is the ground state associated with $H = \omega a^\dagger a$.
Prove that

$$: e^A :: e^B := e^{A+B} : e^{\langle 0 | AB | 0 \rangle} \quad (8)$$

$$e^A e^B =: e^{A+B} : e^{\langle 0 | AB + \frac{A^2}{2} + \frac{B^2}{2} | 0 \rangle}. \quad (9)$$

Then calculate $\langle 0 | e^A e^B | 0 \rangle$.

Problem 6. Gauge equivalence

Consider a one dimensional system of length L_x with periodical boundary condition, i.e. $\psi(x + L_x) = \psi(x)$. The system is subject to a constant vector potential A_x , and define $\phi = A_x L_x$.

$$H = \frac{1}{2m} \left(P - \frac{e}{c} A_x \right)^2 + V(x), \quad (10)$$

1) Solve the single states at $V = 0$ with the flux ϕ . Check as $\phi \rightarrow \phi + \phi_0$ how do these states evolve? $\phi_0 = hc/e$ is the fundamental flux quanta. Calculate the electric current I_j contributing from the state ψ_j and show that $I_j = -c \frac{\partial E_j}{\partial \phi}$.

2) Consider $H = H_0 - \gamma \delta(x - L/2)$ with $\gamma > 0$. Solve the bound state in the limit of $\hbar^2/(m\gamma) \ll L$. How does this state evolve as $\phi \rightarrow \phi + \phi_0$.

Problem 7. Dirac and Weyl Hamiltonians

Read my lecture note 9, and go through the following calculations.

1) Consider the Hamiltonian

$$H_{2D} = p_x \sigma_y - p_y \sigma_x \quad (11)$$

Solve the plane wave states with the 2D momentum (p_x, p_y) .

2) Consider the Hamiltonian

$$H_{3D} = p_x \sigma_x + p_y \sigma_y + p_z \sigma_z \quad (12)$$

Solve the plane wave states with the 3D momentum (p_x, p_y, p_z) .

3) Consider the Hamiltonian

$$H_{3D} = (\mathbf{p} - \frac{e}{c} \mathbf{A}) \cdot \boldsymbol{\sigma}, \quad (13)$$

with $\mathbf{A} = (0, Bx, 0)$. Solve its energy spectra and plot them as a function of k_z .