

Quantum Mechanics 2026 HW4

Due 9/30 in Class

September 24, 2025

Problem 1. Probability current

Consider the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t) \right) \psi(\mathbf{r}, t). \quad (1)$$

Define the probability density

$$\rho(x, t) = \psi^\dagger(x, t) \psi(x, t).$$

In order to have the probability conservation, i.e.,

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j} = 0,$$

how should we define the probability current $\mathbf{j}(\mathbf{r}, t)$?

Problem 2. Time-evolution

Consider a physical observable O , and its expectation value

$$\bar{O}(t) = \langle \psi(t) | O | \psi(t) \rangle.$$

The Hamiltonian of the system is time-independent.

Prove the following statements:

- 1) If $[O, H] = 0$, then

$$\frac{d}{dt} \bar{O}(t) = 0.$$

- 2) If $|\psi(t)\rangle$ itself is the time-evolution of an eigenstate of H , then

$$\frac{d}{dt} \bar{O}(t) = 0.$$

Problem 3. The f -sum rule

Consider a particle in 1D whose Hamiltonian is given by

$$H = p^2/2m + V(x). \quad (2)$$

By calculating $[[H, x], x]$, prove

$$\sum_l |\langle l | x | n \rangle|^2 (E_l - E_n) = \hbar^2/2m, \quad (3)$$

where $|n\rangle$ and $|l\rangle$ are energy eigenstates of H with the eigenvalues E_n and E_l , respectively. This is essentially the quantization condition that Heisenberg used to establish quantum mechanics.

Problem 4. The double δ -potential

A particle of mass m moves in 1D x under the potential $V(x)$:

$$V(x) = \gamma \left(\delta \left(x + \frac{a}{2} \right) + \delta \left(x - \frac{a}{2} \right) \right). \quad (4)$$

- 1) Consider the scattering states: At $x < -\frac{a}{2}$, it is a superposition of an incident wave superposed with a reflection wave

$$\psi(x) = e^{ikx} + R e^{-ikx}.$$

At $x > \frac{a}{2}$, it is the transmission wave

$$\psi(x) = S e^{ikx}.$$

Find the expression of the scattering amplitudes R and S .

- 2) Please find the conditions for the perfect transmission, i.e., $|S| = 1$.
- 3) Please find the bound state solutions at $\gamma < 0$. Express R and S in terms of E . Please show that the bound state information shows up as the pole of the scattering amplitude.

Problem 5. Bound states

Consider a 1D potential $V(x) \leq 0$ for $-\infty < x < +\infty$. As $|x| \rightarrow +\infty$, $V(x) \rightarrow 0$ and $V(x=0) \neq 0$. Please prove that for the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x), \quad (5)$$

there must exist at least one bound state $\psi_b(x)$ whose eigenenergy $E_b < 0$.

Hint: Consider the following problem. Suppose two potential functions $V_{1,2}(x)$ satisfying

$$V_1(x) < V_2(x)$$

for $-\infty < x < +\infty$, then prove that the ground state energy E_{g1} for the system with potential V_1 is smaller than that of E_{g2} for the system with potential V_2 .