Elementary operator identities of bosonization

operator normal product

$$A = \alpha a + \alpha' a^\dagger \qquad B = \beta a + \beta' a^\dagger$$

if $[A,B]$ commutes with A, B

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} = e^B e^A e^{[A,B]}$$

$$:e^A::e^B: = e^{\alpha' a^\dagger} e^{\alpha a} e^{\beta' a^\dagger} e^{\beta a} = e^{\alpha' a^\dagger} e^{\beta' a^\dagger} e^{\alpha a} e^{\beta a} \, e^{[\alpha a, \beta' a^\dagger]}$$

$$= :e^{A+B}: e^{\langle 0|AB|0\rangle}$$

$$e^A e^B = e^{\alpha' a^\dagger + \alpha a} e^{\beta a^\dagger + \beta a} = e^{\alpha' a^\dagger} e^{\alpha a} e^{\beta' a^\dagger} e^{\beta a} \cdot e^{-\frac{1}{2}[\alpha' a^\dagger, \alpha a] - \frac{1}{2}[\beta' a^\dagger, \beta a]}$$

$$= :e^A::e^B: e^{\frac{1}{2}\alpha\alpha' + \frac{1}{2}\beta\beta'}$$

$$= :e^{A+B}: e^{\langle 0| AB + \frac{A^2}{2} + \frac{B^2}{2} |0\rangle}$$

useful identities

① $e^{-B} e^A e^B = A + [A,B] + \frac{1}{2!}[[A,B],B] + \cdots$

② If $C = [A,B]$ and $[C,A] = [C,B] = 0$, then

$$e^{-B} e^A e^B = A + C, \qquad [A, e^B] = C e^B$$

$\cancel{e^A e^B}$

$$\boxed{\psi_R = \frac{1}{\sqrt{2\pi a}}\, e^{i\sqrt{4\pi}\,\phi_R(x)} \quad , \quad \psi_L = \frac{1}{\sqrt{2\pi a}}\, e^{-i\sqrt{4\pi}\,\phi_L(x)}}$$

$$\phi_R(x) = \frac{Q_0}{2} + \varphi_R(x) + \varphi_R^\dagger(x)$$

convergence factor

$$\varphi_R(x) = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}}\, b_q\, e^{iqx - aq/2} \qquad n_q = \frac{|q|}{2\pi/L}$$

$$\varphi_R^\dagger(x) = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}}\, b_q^\dagger\, e^{-iqx - aq/2}$$

$$\phi_L(x) = \frac{\mathbb{P}_0}{2} + \varphi_L(x) + \varphi_L^\dagger(x) \qquad \text{where } [b_q, b_{q'}^\dagger] = i\delta_{qq'}$$

$$\varphi_L(x) = \sqrt{\frac{1}{4\pi}} \sum_{q<0} \frac{1}{\sqrt{n_q}}\, b_q\, e^{iqx + aq/2}$$

$$\varphi_L^\dagger(x) = \sqrt{\frac{1}{4\pi}} \sum_{q<0} \frac{1}{\sqrt{n_q}}\, b_q^\dagger\, e^{-iqx - aq/2}$$

$\ast$ Commutation relations (I)

$$\boxed{\begin{aligned} [\varphi_R(x), \varphi_R^\dagger(x')] &= \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a - i(x-x')) \\ [\varphi_L(x), \varphi_L^\dagger(x')] &= \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a + i(x-x')) \end{aligned}}$$

Proof: $[\varphi_R(x), \varphi_R^\dagger(x')] = \frac{1}{4\pi} \sum_{q>0} \frac{1}{n_q}\, e^{iq(x-x') - aq}$

$$= \frac{1}{4\pi} \sum_{n_q=1}^{\infty} \frac{1}{n_q}\, e^{i\frac{2\pi}{L}[(x-x') - a]n_q}$$

$$= \frac{-1}{4\pi} \ln\left[1 - e^{i\frac{2\pi}{L}\{(x-x') - a\}}\right] \xrightarrow[L\to+\infty]{} = \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a - i(x-x'))$$

$$[\varphi_L(x), \varphi_L^{\dagger}(x')] = \frac{1}{4\pi} \sum_{q<0} \frac{1}{n_q} e^{iq(x-x')-aq}$$

$$= \frac{1}{4\pi} \sum_{n_q=1}^{\infty} \frac{1}{n_q} \left(e^{-i\frac{2\pi}{L}(x-x')-a}\right)^{n_q} = \frac{-1}{4\pi} \ln\left(1 - e^{-i\frac{2\pi}{L}(x-x')-a}\right]$$

$$= \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a + i(x-x'))$$

* commutation relations (II)

$$[\phi_R(x), \phi_R(x')] = [\varphi_R(x) + \varphi_R^{\dagger}(x), \varphi_R(x') + \varphi_R^{\dagger}(x')]$$

$$= [\varphi_R(x), \varphi_R^{\dagger}(x')] + [\varphi_R^{\dagger}(x), \varphi_R(x')]$$

$$= \frac{-1}{4\pi} \ln \frac{2\pi}{L}[a - i(x-x')] - \left(\frac{-1}{4\pi}\right) \ln \frac{2\pi}{L}(a + i(x-x'))$$

$$= \frac{-1}{4\pi} \ln \frac{a - i(x-x')}{a + i(x-x')} = \frac{i}{4} \operatorname{sgn}(x-x')$$

if $(x-x') > 0 \Rightarrow \frac{-1}{4\pi}\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) i = i/4$

$x - x' < 0 \qquad \frac{-1}{4\pi}\left(\frac{\pi}{2} \times 2\right) i = -i/4$

$$[\phi_L(x), \phi_L(x')] = [\varphi_L(x), \varphi_L^{\dagger}(x')] + [\varphi_L^{\dagger}(x), \varphi_L(x')]$$

$$= \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a + i(x-x')) - \left(\frac{-1}{4\pi}\right) \ln \frac{2\pi}{L}(a - i(x-x'))$$

$$= \frac{-1}{4\pi} \ln \frac{a + i(x-x')}{a - i(x-x')} = -\frac{i}{4} \operatorname{sgn}(x-x')$$

$$[\phi_R(x),\ \phi_L(x')] = \frac{1}{4}[Q_0,\ \mathbb{P}_0] = i/4$$

Summary:
$$\begin{cases} [\phi_R(x),\ \phi_R(x')] = i/4\ \mathrm{sgn}(x-x') \\ [\phi_L(x),\ \phi_L(x')] = i/4\ \mathrm{sgn}(x-x') \\ [\phi_R(x),\ \phi_L(x')] = i/4 \end{cases}$$

★ Check anti-commutation relations (III)

$$\psi_R(x)\ \psi_R(x') = \frac{1}{2\pi a}\, e^{i\sqrt{4\pi}\,\phi_R(x)}\ e^{i\sqrt{4\pi}\,\phi_R(x')}$$

$$= \frac{1}{2\pi a}\, e^{i\sqrt{4\pi}\,(\phi_R(x)+\phi_R(x'))} \cdot e^{\frac{1}{2}(-)\cdot 4\pi\,[\phi_R(x),\ \phi_R(x')]}$$

$$= \frac{1}{2\pi a}\, e^{i\sqrt{4\pi}\,(\phi_R(x)+\phi_R(x'))}\ e^{-\frac{i\pi}{2}\mathrm{sgn}(x-x')}$$

$$= \frac{1}{2\pi a}\, e^{i\sqrt{4\pi}\,[\phi_R(x'),\ \phi_R(x)]}\ e^{-i\frac{\pi}{2}\mathrm{sgn}(x'-x)}\ e^{\pm i\pi}$$

$$= -\ \psi_R(x')\ \psi_R(x)$$

Similarly $\psi_L(x)\,\psi_L(x') = -\ \psi_L(x')\,\psi_L(x)$.

$$\psi_R(x)\ \psi_L(x') = \frac{1}{2\pi a}\, e^{i\sqrt{4\pi}\,\phi_R(x)}\ e^{-i\sqrt{4\pi}\,\phi_L(x')}$$

$$= \frac{1}{2\pi a}\, e^{i\sqrt{4\pi}\,(\phi_R(x)-\phi_L(x'))}\ e^{\frac{1}{2}\cdot 4\pi\cdot i/4}$$

$$= \frac{1}{2\pi a}\, e^{i\sqrt{4\pi}\,(-\phi_L(x')+\phi_R(x))}\ e^{-\frac{1}{2}\cdot 4\pi\cdot i/4}$$

$$= -\ \psi_L(x')\ \psi_R(x)$$

Define $\phi = \phi_R + \phi_L$, $\quad \theta = \phi_R - \phi_L$

$$[\phi(x), \theta(x')] = [\phi_R(x) + \phi_L(x), \phi_R(x') - \phi_L(x')]$$

$$= [\phi_R(x), \phi_R(x')] - [\phi_L(x), \phi_L(x')] - [\phi_R(x), \phi_L(x')] + [\phi_L(x), \phi_R(x')]$$

$$= \frac{i}{2}[\mathrm{sgn}(x-x') - 1] = -i\,\Theta(x'-x)$$

$$[\phi(x), \partial_x\theta(x')] = -i\delta(x-x'). \Rightarrow \begin{cases} \partial_x\theta(x) = -\Pi_\phi \\ \Pi_\phi = \dfrac{\partial\mathcal{L}}{\partial\dot{\phi}} \end{cases}$$

(*) Vertex operator $\quad e^{i\beta\phi_R(x)}, \; e^{i\beta\phi_L(x)}, \; e^{i\beta\phi(x)}$

$$\boxed{\langle G| e^{i\beta\phi_R(x)} e^{-i\beta'\phi_R(0)} |G\rangle = \begin{cases} \left[\dfrac{a}{a-ix}\right]^{\frac{\beta^2}{4\pi}} & (\beta=\beta') \\ 0 & (\beta\neq\beta') \end{cases}}$$

Proof: $\quad e^{i\beta\phi_R(x)} e^{-i\beta'\phi_R(0)} = :e^{i\beta\phi_R(x) - i\beta'\phi_R(0)}:\, e^{\langle G|\beta\beta'\phi_R(x)\phi_R(0) - (\beta^2+\beta'^2)\phi(0)|G\rangle}$

$$e^{\langle G|\beta\beta'(\phi_R(x,t)\phi_R(0) - \phi_R^2(0))|G\rangle} \quad e^{-\frac{(\beta-\beta')^2}{2}\langle G|\phi_R^2(0)|G\rangle} = ?$$

$$\langle G|\phi_R(x)\phi_R(0)|G\rangle = \langle G|\varphi_R(x)\varphi_R^\dagger(0)|G\rangle$$

$$= \langle G|[\varphi_R(x), \varphi_R^\dagger(0)]|G\rangle = \frac{-1}{4\pi}\ln\frac{2\pi}{L}(a-ix)$$

$$\langle G|\phi_L(x)\phi_L(0)|G\rangle = \langle G|[\varphi_L(x), \varphi_L^\dagger(0)]|G\rangle = \frac{-1}{4\pi}\ln\frac{2\pi}{L}(a+ix)$$

$$\langle G|\phi_R^2(0)|G\rangle = \frac{-1}{4\pi}\ln\frac{2\pi a}{L}$$

$$e^{-\frac{(\beta-\beta')^2}{2}\langle G|\phi_R^2|G\rangle} = e^{-\frac{(\beta-\beta')^2}{8\pi}\ln\frac{L}{2\pi a}} = \left(\frac{L}{2\pi a}\right)^{-\frac{(\beta-\beta')^2}{8\pi}}$$

$$\xrightarrow{\text{if } \beta\neq\beta'} 0$$

$$e^{\beta^2\langle G|\phi_R(x)\phi_R(0)-\phi_R^2(0)|G\rangle} = e^{-\frac{\beta^2}{4\pi}\ln\frac{a-ix}{a}} = \left(\frac{a}{a-ix}\right)^{\frac{\beta^2}{4\pi}}$$

Hence,

$$\boxed{\langle G|e^{i\beta\phi_R(x)}e^{-i\beta'\phi_R(0)}|G\rangle = \delta_{\beta\beta'}\left(\frac{a}{a-ix}\right)^{\frac{\beta^2}{4\pi}}}$$

Similarly

$$\boxed{\langle G|e^{i\beta\phi_L(x)}e^{-i\beta\phi_L(0)}|G\rangle = \delta_{\beta\beta'}\left(\frac{a}{a+ix}\right)^{\frac{\beta^2}{4\pi}}}$$

$$\langle G|e^{i\beta\phi(x,t)}e^{-i\beta\phi(0)}|G\rangle = \left(\frac{a}{a-ix}\right)^{\frac{\beta^2}{4\pi}}\left(\frac{a}{a+ix}\right)^{\beta^2/4\pi}$$

Proof:

$$e^{i\beta\phi(xt)}e^{-i\beta\phi(0)} = :e^{-i\beta(\phi(x,t)\phi(x,0)-\phi^2(0))}:\ e^{\langle G|\beta^2(\phi(xt)\phi(0)-\phi^2(0))|G\rangle}$$

$$\langle G|\tilde\phi(xt)\tilde\phi(0)-\tilde\phi^2(0)|G\rangle = \langle G|(\phi_R(x)+\phi_L(x))(\phi_R(0)+\phi_L(0)) - (\phi_R+\phi_L)^2|G\rangle$$

$$= \langle G|\phi_R(x)\phi_R(0)-\phi_R^2(0)|G\rangle + \langle G|\phi_L(x)\phi_L(0)-\phi_L^2(0)|G\rangle$$

Hence

$$\boxed{\langle G|e^{i\beta\phi(xt)}e^{-i\beta\phi(0)}|G\rangle = \left(\frac{a^2}{a^2+x^2}\right)^{\frac{\beta^2}{4\pi}}}$$

(If Luttiger parameter $K\neq 1$, then the power $\rightarrow \frac{\beta^2 K}{4\pi}$).

So far we considered the case of $K=1$.