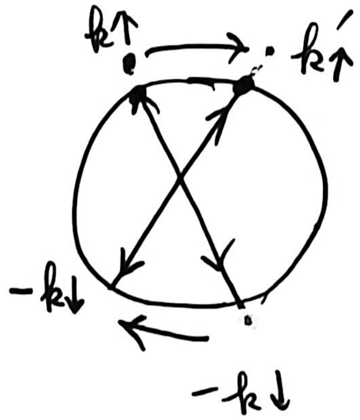
Lecture 1 Overview of Superconductivity

1 Phenomenological theory – London Eq, GL Eq

2. Cooper problem

3 BCS trial wavefunction



$$\psi = \prod_k (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|\Omega\rangle$$

$$\Delta_k = \int \frac{dk'}{(2\pi)^3}\, g(k,k')\, \frac{\Delta_{k'}}{2E_{k'}} \tanh \frac{\beta}{2} E_{k'}$$

$$\Delta / k_B T_c \simeq 1.76 .$$

4. McMillian formula

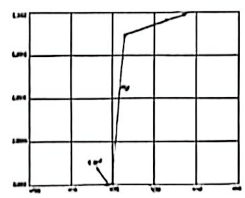
$$T_c \simeq \hbar\omega_D \exp\left[-\frac{1}{N_0 \bar{V}_{ph} - \mu^*}\right], \quad \mu^* = \frac{N_0 V_c}{1 + N_0 V_c \ln \frac{\Lambda}{\omega_D}}$$

5. phase fluctuation $\dfrac{T_\theta}{\Delta} \simeq \dfrac{1}{\alpha} \dfrac{c}{v_F} \left(\dfrac{\xi}{\lambda}\right)^2 .$

- 1911 Onnes, discovery of Hg superconductivity at 4.2K
- 1937 Kapitsa, Allen, Misener's discovery of superfluid ^{4}He

- Feynman could not solve it!

"The story of Feynman's epic battle with superconductivity is a little-known chapter in his scientific career. As it turned out, Feynman was the loser in that struggle……"



$1\times 10^{-6}\,\Omega$

Zero resistance

§ Importance of superconductivity

1 W. Pauli, Solid state physics is "Schmutzphysik"

2. P.W. Anderson, "More is different"

Amazing and puzzling factors that cannot be understood in the single-electron history.

1. What protects the zero resistivity?

2. Why does a superconductor exihit the complete diamagnetism?

3. best metals are poor superconductors or non-superconducting

Superconductivity is not a property of one or a few electrons but emerges as a collective behavior of a society of electrons

New principle appears: off-diagonal long-range corner.

phase coherence, Anderson-Higgs mechanism →

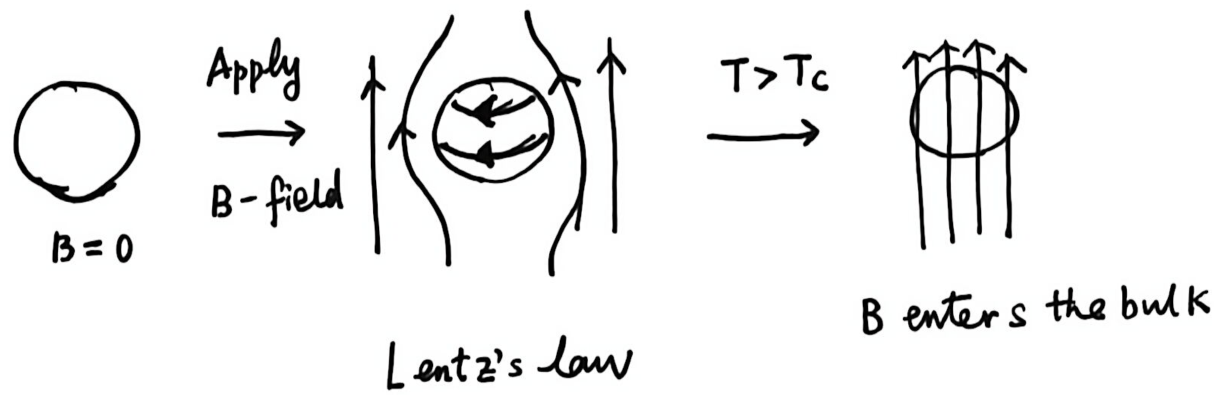
Entire field of condensed matter physics including high energy

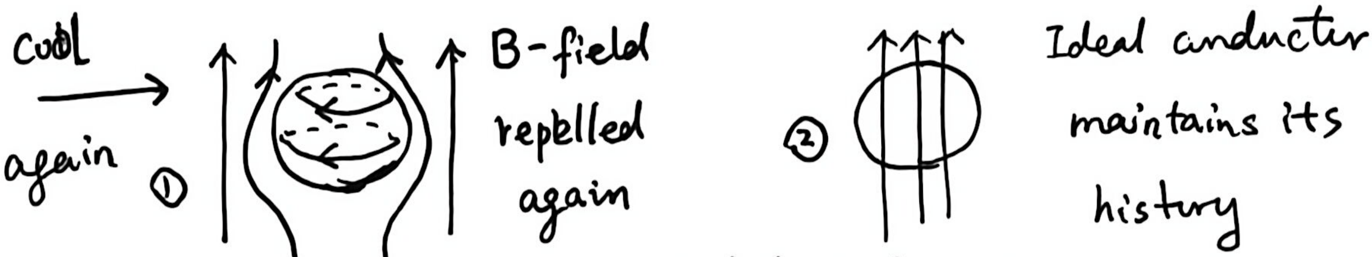
1 A brief history of superconductivity

1. 1911 Onnes' discovery of superconductivity of Hg 4.2K $R: 0.1\Omega \to 10^{-6}\Omega$
2. 1933 Meissner-Ochsenfeld effect
3. 1935 London equation
4. 1937 Kapitsa, Allen, Misener's discovery of superfluid ^{4}He
5. 1950 Ginzburg-Landau theory
6. 1957 Abrikosov's vortex state
7. 1957 The microscopic theory – BCS
8. 1962 Josephson effect
9. 1962 Anderson-Higgs mechanism
10. 1971 ^{3}He superfluidity
11. 1986 Bednorz and Mueller's discovery of high T_c superconductivity
12. 2006 Iron-based superconductivity
13. 2010's Topological superconductivity, Majorana fermion
14. 2010's Superconductivity of hydrides, H_2S, LaH_{10}

2. Diamagnetism – beyond Lentz law

$T < T_c$ Apply B-field





screening current's direction against Lentz law (Hirsch)

3. London equation:

superfluid component of electrons.

$$\frac{d\vec{v}_s}{dt} = \frac{\partial \vec{v}_s}{\partial t} + (\vec{v}_s \cdot \nabla)\vec{v}_s = \frac{e}{m}(\vec{E} + \frac{1}{c}\vec{v}_s \times \vec{B})$$

$$\frac{\partial \vec{v}_s}{\partial t} - \frac{e}{m}\vec{E} + \frac{1}{2}\nabla v_s^2 = \vec{v}_s \times (\nabla \times \vec{v}_s + \frac{e}{mc}\vec{B})$$

$$\nabla\times \quad \rightarrow \quad \frac{\partial}{\partial t}(\nabla \times \vec{v}_s + \frac{e}{mc}\vec{B}) = \nabla \times (\vec{v}_s \times (\nabla \times \vec{v}_s + \frac{e}{mc}\vec{B}))$$

a stronger assumption: at $t=0$

$$\nabla \times \vec{v}_s + \frac{e}{mc}\vec{B} = \nabla \times (\vec{v}_s + \frac{e}{mc}\vec{A}) = 0$$

then it will remain zero for all the time: F. London and H. London

Coulomb gauge:

$$\vec{v}_s = \frac{1}{m}(\hbar\nabla\phi - \frac{e}{c}\vec{A}) \quad \leftarrow \quad \vec{J}_s = e n_s v_s$$

$$\vec{J}_s = -\frac{e^2 n_s}{mc}\vec{A} = -\frac{c}{4\pi\lambda_L^2}\vec{A}$$

where λ_L is a length scale satisfying $\frac{1}{\lambda_L^2} = \frac{4\pi n_s e^2}{mc^2} = \frac{\omega_p^2}{c^2}$

ω_p: plasma frequency of the superfluid component.

λ_L: London penetration depth!

$$\frac{n_s^*}{m^*}e^{*2} = \frac{n_s}{m}e^2$$

$e^* = 2e \quad n_s^* = \frac{n_s}{2}$

$m^* = 2m$

: Why is it remarkable?

normal metal: $\vec{A}(\vec{q}) = \vec{A}_L(\vec{q}) + \vec{A}_T(\vec{q})$

where $\vec{A}_L \parallel \vec{q}$ and $\vec{A}_T \perp \vec{q}$. $\Rightarrow$
$$\begin{cases} A_{L,i} = \dfrac{q_i q_j}{q^2} A_j \\ A_{T,i} = \left(\delta_{ij} - \dfrac{q_i q_j}{q^2}\right) A_j \end{cases}$$

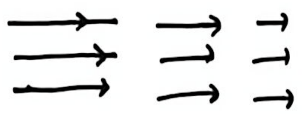
$\vec{A}_L$ is a pure gauge

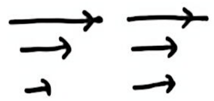
which should not have any observable effects

$$J^L(\vec{q}) = \chi^L_{JJ}(\vec{q},0)\,\vec{A}_L(q) = 0$$, where χ_{JJ} is the current-current response

Gauge invariance ensures that $\chi^L_{JJ}(\vec{q},0) = 0$.

On the other hand, as $q \to 0$, there are no difference between A_L and A_T


$\vec{A}_L$


$\vec{A}_T$

$$\lim_{q\to 0} \chi^T_{JJ}(q, \omega=0)$$
$$= \chi^L_{JJ}(q,0) = 0$$

However, the superconducting state does see the difference

$\chi^{qL}_{JJ} = 0 \leftarrow$ gauge invariance. $\quad \vec{q} \times \vec{J}_s(q) = -\dfrac{n_s e^2}{mc}\, \vec{q} \times \vec{A}(q)$

$\Rightarrow$ transverse field $\quad \chi^T_{JJ}(q,0) = -\dfrac{n_s e^2}{mc}$

$$\Rightarrow \lim_{q\to 0}\left(\chi^T_{JJ}(q,0) - \chi^L_{JJ}(q,0)\right) \neq 0.$$

Even in the long-wave length limit, the system still distinguish the trasverse and longitudinal $\vec{A}$ field! $\rightarrow$ A consequence of long-range phase coherence.

§ Penetration depth

$$\nabla\times(\nabla\times\vec{J}_s) = -\frac{n_s e^2}{mc}\nabla\times\vec{B} = -\frac{4\pi n_s e^2}{mc^2} J_s$$

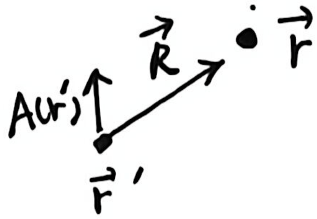
$$\cancel{\nabla(\nabla\cdot\vec{J}_s)} - \nabla^2 J_s = -\lambda_L^{-2} J_s \Rightarrow J_s, B \text{ decay as } e^{-x/\lambda_L}$$

§ Pippard non-local form - coherence length §

Chambers formula of AC responce of normal metal

$$\vec{J}(\vec{r},\omega) = \frac{e^2 v_f}{4\pi}\frac{\partial n}{\partial \varepsilon}\int d\vec{r}'\, \frac{(\vec{E}(r')\cdot\vec{R})\,\vec{R}}{R^4}\, e^{-i\frac{\omega R}{v_f} - R/l}$$

$$\rightarrow \vec{J}(\vec{r}) = C\int d\vec{r}'\, \frac{(\vec{A}(r')\cdot\vec{R})}{R^2}\, e^{-R/\xi_0}$$



ξ_0 is a length scale - Pippard correlation length

microscopically, it can be derived as $\xi_0 = \hbar v_f / \pi\Delta$

where Δ is the zero ~~temperature~~ gap function

If A is slow varying $\rightarrow$ London equation, $\vec{A} \parallel \hat{z}$

$$\vec{J}(\vec{r}) = C A \int \frac{\cos^2\theta}{R^2} e^{-R/\xi_0} R^2 dR \sin\theta \, d\theta \, d\phi = \frac{4\pi}{3} C \xi_0 A$$

$$= -\frac{n_s e^2}{mc} A \quad \Rightarrow \quad C = -\frac{3}{4\pi} \frac{n_s e^2}{mc\xi_0}$$

- In the dirty limit, mean-free path l

$$\vec{J}(r) = C \int dr' \frac{(\vec{A}(r') \cdot \vec{R}) \vec{R}}{R^4} e^{-R/\xi_0 - R/l}$$

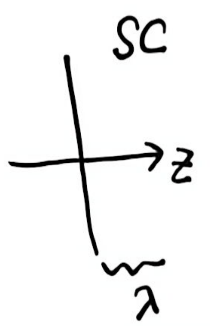
$$\Rightarrow \vec{J}(r) = \frac{1}{\lambda_L^2} \frac{l}{\xi_0 + l} \vec{A}(r) \longrightarrow \frac{l}{\xi_0 \lambda_L^2} A(r)$$

$$\Rightarrow \lambda = \lambda_L (\xi_0 / l)^{1/2} \qquad (\lambda \gg l, \ \xi_0 \gg l)$$

☆ Disorder suppresses the superfluid density and increases the penetration depth.

- Pippard limit $\xi_0 \gg \lambda_L$ (London limit $\lambda_L \gg \xi_0$)

$\vec{A}$ is only non-zero within a length of λ



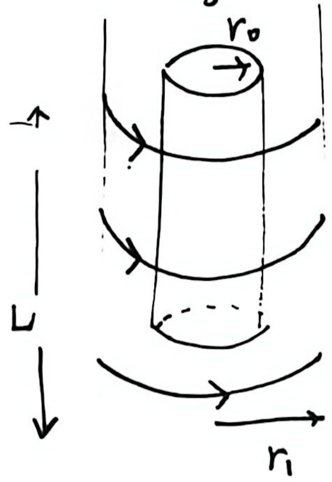
$$\vec{J} = -\frac{n_s e^2}{mc} \frac{\lambda}{\xi_0} \vec{A}$$

$$\lambda^{-2} = \frac{\lambda}{\xi_0} \lambda_L^{-2} \Rightarrow \lambda = (\xi_0 \lambda_L^2)^{1/3}$$

$$\frac{\lambda}{\lambda_L} = \left(\frac{\xi_0}{\lambda_L}\right)^{1/3}$$

	λ_L	ξ_0	λ
Al	157	16000	530
Sn	355	2300	560
Pb	370	830	480

Thermodynamics of SC



long cylinder : sample in a solenoid

$$H = \frac{4\pi N I}{c L} \quad N:\ \text{number of turns}$$

normal state: $F_n = \pi r_0^2 L f_n + \pi r_1^2 L \frac{H^2}{8\pi}$

superconducting state $F_s = \pi r_0^2 L f_s + \pi (r_1^2 - r_0^2) L \cdot \frac{H^2}{8\pi}$

in order to repel the flux, the work did ~~by~~ $\frac{\text{emf}}{\text{induced}}$

$$\int V I\, dt = \int_i^f -\frac{N}{c}\left(\frac{d\phi}{dt}\right) I\, dt = -\frac{N}{c}(\phi_f - \phi_i) I = \frac{N}{c} I \pi r_0^2 H$$

$$= \pi r_0^2 L \frac{H^2}{4\pi}$$

$\Rightarrow$ at $H_c(T)$, we have equalibrium $\Rightarrow$ $f_n = f_s - \frac{H^2}{8\pi} + \frac{H^2}{4\pi}$

$$= f_s + \frac{H^2}{8\pi}$$

another way to derive it is to go through

the Gibbs free energy $G = F - MH$

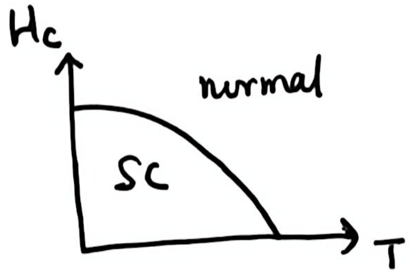
at H_c we have $G_s(H_c) = G_n(H_c)$

The normal state G doesn't depend on H much $G_n(H_c) \approx f_n(0)$.

in SC state $dG = -M\, dH = \frac{H\, dH}{4\pi} \quad \left(M = \frac{-H}{4\pi}\right)$

$$G_s(H) - G_s(0) = \int \frac{H\, dH}{4\pi} = \frac{H^2}{8\pi}$$

$$\Rightarrow G_s(H) = f_s(0) + \frac{H^2}{8\pi} \Rightarrow \boxed{f_s(0) + \frac{H^2}{8\pi} = f_n(0)}$$



§ G L free energy

What's the order parameter $\psi(r)$ for a superconductor?

① $\psi(r)$ should be complex, such that it couples to EM

② normalization $|\psi(r)|^2 \propto n_s(r)$

$$f(\psi, A) = \frac{1}{2m^*} \Big|\underbrace{\left(-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)}_{\hat{\psi}}\Big|^2 + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{(\nabla\times A)^2}{8\pi}$$

$$\alpha = \alpha_0\left(\frac{T}{T_c} - 1\right)$$

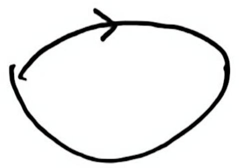
$$G = \int d^3r \left(f(\psi, A) - \frac{1}{4\pi}\vec{B}\cdot\vec{H}\right) = \int d^3r \left[f(\psi, A) - \frac{1}{4\pi}(\nabla\times A)\cdot H\right]$$

$$\delta G/\delta\psi^* \Rightarrow \frac{-1}{2m^*}\left(-i\hbar\nabla - \frac{e^*}{c}A\right)^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0$$

Flux quantization : $\psi(r) = \sqrt{\rho_s(r)}\, e^{i\varphi(r)}$

$$\cancel{J_s(r) = 0} \Rightarrow \quad J_s = \frac{e^*\hbar}{m^*}\rho_s(r)\left(\nabla\phi(r) - \frac{e^*}{\hbar c}A(r)\right)$$

$$J_s(r) = 0 \quad \Rightarrow \oint dr\left[\nabla\phi(r) - \frac{e^*}{\hbar c}A(r)\right] = 0$$



$$2n\pi = \frac{e^*}{\hbar c}\oint dr\, A(r) \Rightarrow \Phi = \frac{2n\pi}{e^*/\hbar c}$$

$$\Rightarrow \Phi = n\cdot\frac{hc}{e^*} = n\frac{hc}{2e}$$

$e^* = 2e$, $m^* = 2m$, $\rho_s(r) = n_s/2$ $\rightarrow$ London Eq.

$$\frac{c}{4\pi\lambda_L^2} = \frac{e^{*2}}{m^* c}\rho_s \rightarrow \boxed{\lambda_L^2(T) = \frac{m^* c^2}{4\pi e^{*2}}\frac{\beta}{|\alpha|}} \propto T_c - T$$

$$\rho_s = |\psi_0|^2 = \frac{|\alpha|}{\beta} \propto T_c - T$$

healing length $\boxed{\xi^2(T) = \frac{\hbar^2}{2m^*|\alpha|}} \propto |T - T_c|$ $\rightarrow \xi(T) \propto (T-T_c)^{1/2}$

$\exists$ critical exponent (~~MF~~) mean field $\beta = 1/2$ $\nu = 1/2$

~~Minimizing free~~

$$f_s(H=0) - f_n(H=0) = \frac{H_c^2}{8\pi} = \frac{\alpha^2}{2|\beta|} \Rightarrow \frac{H_c^2}{8\pi} = \frac{\alpha^2}{|\beta|}$$

$$\frac{1}{\lambda_L(T)\xi(T)} = \frac{e^*}{\hbar c}\left(\frac{8\pi|\alpha|^2}{\beta}\right)^{1/2} = \frac{\sqrt{2}e^*}{\hbar c} H_c$$

$$H_c(T) = \frac{\Phi_0}{2\pi\sqrt{2}\,\xi(T)\lambda(T)}$$

2.3 Upper critical field H_{c2}

We consider a critical field at which superconductivity is completely suppressed. Such a field is called H_{c2}. Later on, we will see that it actually happens when the vortex cores touch each other. In this case, H_{c2} is at the order of Φ_0/ξ^2. Now we start with GL equation. Since at H_{c2}, the order parameter nearly is nearly zero, we can neglect the $|\Psi|^4$ term, and arrive at

$$f_s = -\frac{\hbar^2}{2m^*}\Psi^*\Big(-i\hbar\nabla - \frac{e^*}{c}\mathbf{A}(\mathbf{r})\Big)^2\Psi + \alpha\Psi^*\Psi. \tag{20}$$

The cyclotron frequency $\omega_c = \frac{eB}{m*c}$, which set up the Landau level gap $\hbar\omega$. The zero point motion energy is $\frac{1}{2}\hbar\omega_c$. Hence, when

$$\begin{aligned} \frac{1}{2}\hbar\omega_c &= |\alpha|, \Rightarrow \frac{\hbar\, e^* B}{2\, m^* c} = |\alpha|, \Rightarrow \frac{\hbar^2}{2m^*|\alpha|} = \frac{\hbar c}{e^* B} \\ B &\approx H_{c2} = \frac{hc/e*}{2\pi\xi^2} = \frac{\Phi_0}{2\pi\xi^2}. \end{aligned} \tag{21}$$

2.4 The GL parameter κ: type I and II

Now we can define a dimensionless quantity,

$$\begin{aligned} \kappa &= \frac{\lambda_L(T)}{\xi(T)} = \Big(\frac{\beta}{2\pi}\Big)^{\frac{1}{2}}\frac{m^* c}{\hbar e^*} = \frac{\lambda_L^2(T)}{\lambda_L(T)\xi(T)} \\ \kappa &= \sqrt{2}H_c\lambda_L^2(T)/(\frac{\hbar c}{e^*}) = 2\sqrt{2}\pi\frac{H_c\lambda_L^2(T)}{\Phi_0}, \end{aligned} \tag{22}$$

which is called the Ginzburg-Landau parameter. κ plays an important role in determining type I and II superconductors.

It is easy to check that

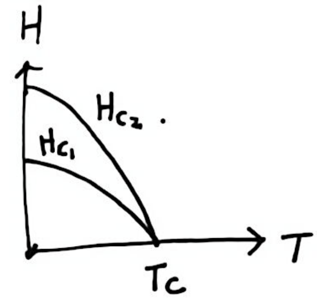
$$\frac{H_{c2}}{H_c} = \sqrt{2}\kappa. \tag{23}$$

We will have the following two situations, denoted as the type (II) and (I) superconductors, respectively.

1. Type II: If $\kappa > \frac{1}{\sqrt{2}}$, then $H_{c2} \geq H_c$. When $H_{c2} > H > H_c$, we still have superconducting condensation, since H is not enough to kill Ψ completely. Nevertheless, it cannot completely expel the magnetic flux either. Otherwise, the Meissener phase will not be energetically favorable at $H > H_c$. Hence, magnetic flux will partly enter the bulk, and it is called the mixed state. Later on, we will that it corresponds to form vortices.

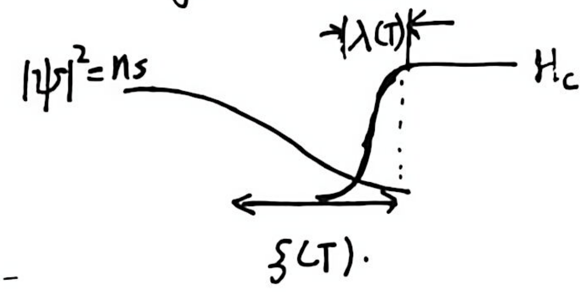
2. Type I: If $\kappa < \frac{1}{\sqrt{2}}$, then $H_c > H_{c2}$. If we lower down the magnetic field, it will first reach H_c, then it becomes the complete Meissner phase, and it will not change as further lowering the field.

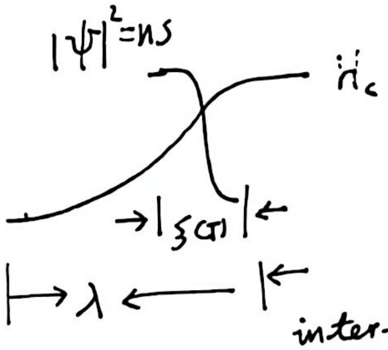
§ type II - superconductor H_{c1} and H_{c2}

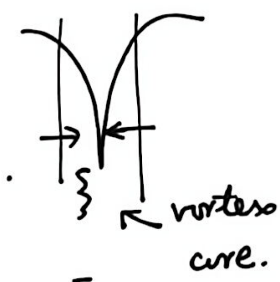


Between H_{c1} and H_{c2}, vortex-state.

Surface energy.







interface - type II

interface - type I

interface : domain between SC / normal face allows magnetic field to enter, thus reduces diamagnetic energy, but it suppresses superconductivity and cost energy.

for type I - superconductor, interface energy > 0, $(\xi \gg \lambda)$.

II ... < 0 $(\xi \ll \lambda)$.

$\longrightarrow$ form vortex lattice with each vortex carries Φ_0, with a normal core with size of ξ.

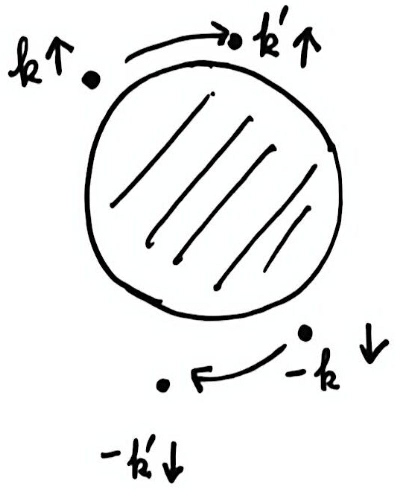
$$H_{c1} = \frac{\Phi}{\lambda_L^2} , \quad H_{c2} = \frac{\Phi}{\xi^2} .$$

§ Cooper pairing — 2 electron problem

put 2-electrons on the top of Fermi surface. — bound state formation. Fermi surface gives rise to a const density of state: Effectively reducing DOS to be 2D like — infinitismall attraction leads to bound state formation. The binding energy

$\boxed{\Delta E = 2\Delta = -2\hbar\omega_D \, e^{-\frac{2}{N_F g}}}$, g is the interaction strength.

We assume $H = \sum_k (\varepsilon_k - \mu) C^+_{k\sigma} C_{k\sigma} - g \sum_{kk'} C^+_{k\uparrow} C^+_{-k\downarrow} C_{-k'\downarrow} C_{k'\uparrow}$



$\Delta E \propto e^{-\frac{2}{N_F g}}$ ← intrinsic singularity

cannot be expanded interm of power series! — Failure of Feynmann interactions cause scattering $(k\uparrow; -k\downarrow) \longrightarrow (k'\uparrow; -k\downarrow)$.

the eigenstate should be a linear superposition of these states.

$$|\psi\rangle = \sum_k \alpha(k)\, C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle$$

where $\alpha(k)$ is the coefficent, $|F\rangle$ the Full-filled Fermi sphere.

$$H|\psi\rangle = \sum_k \alpha(k) \left(H_0 + \mathcal{U} \right) C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle$$

$$H_0 C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle = \cancel{\left(C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} H_0 |F\rangle + 2\varepsilon_k \right)} = (2\varepsilon_k + E_0)\, C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle$$

$$g \cdot C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle = g \sum_{k'k''} C^\dagger_{k'\uparrow} C^\dagger_{-k'\downarrow} C_{-k''\downarrow} C_{k''\uparrow} C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle$$

$$= g \sum_{k'k''} \delta(k, k'')\, C^\dagger_{k'\uparrow} C^\dagger_{-k'\downarrow} |F\rangle = g \sum_{k'} C^\dagger_{k'\uparrow} C^\dagger_{-k'\downarrow} |F\rangle .$$

$$\Rightarrow H|\psi\rangle = \sum_k \alpha(k) \left[2\varepsilon_k + E_0 \right] C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle - \sum_k \overset{\alpha(k)}{g} \sum_{k'} C^\dagger_{k'\uparrow} C^\dagger_{-k'\downarrow} |F\rangle$$

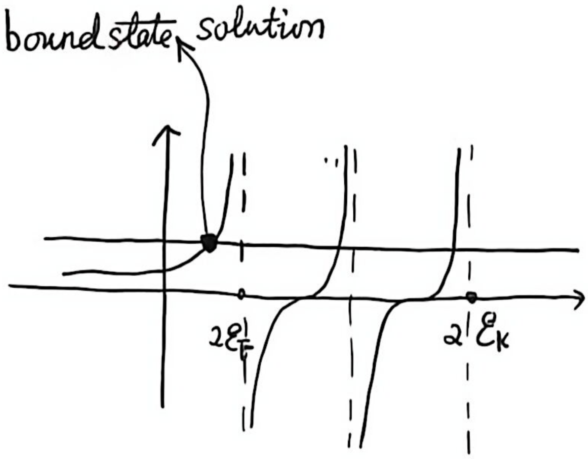
$$= E \sum_k \alpha(k)\, C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle$$

$$\Rightarrow \sum_k \left[(2\varepsilon_k + E_0)\,\alpha(k) - g \sum_{k'} \alpha(k') \right] C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle = E \sum_k \alpha(k)\, C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} |F\rangle$$

i.e. $$(2\varepsilon_k + E_0)\,\alpha(k) - g \sum_{k'} \alpha(k') = E\,\alpha(k)$$

$$\Rightarrow \alpha(k) = \frac{g}{E_0 + 2\varepsilon_k - E} \sum_{k'} \alpha(k')$$

$$\Rightarrow \frac{1}{g} = \sum_k \frac{1}{2\varepsilon_k - (E - E_0)} \Rightarrow +\frac{1}{g} = \sum_k \frac{1}{-\Delta E + 2\varepsilon_k}$$



Cooper pair binding energy!

$$\frac{1}{g} = N(0)\int_0^{\hbar\omega_D} d\epsilon \, \frac{1}{2\mathcal{E} - \Delta E}$$

$$= \frac{N(0)}{2} \ln \frac{2\hbar\omega_D - \Delta E}{-\Delta E}$$

$$\Rightarrow \frac{2}{N(0)\, g} \simeq \ln \frac{2\hbar\omega_D}{|\Delta E|}$$

$$\boxed{\Delta E = 2\hbar\omega_D \, e^{-\frac{2}{N(0)\, g}}}$$

Since one pair of electron form a bound state to gain energy, why not more and more pairs form bound states? — Fermi surface instability!

§ BCS trival WF ← biuld in coherence

Most important discoveries in CMP were originally done by variational wavefunctions.

$$H = \sum_{k\sigma} \xi_{k\sigma} C^{\dagger}_{k\sigma} C_{k\sigma} - \frac{1}{V} \sum_{\substack{k_1 k_2 \\ \sigma\sigma'}} g(k_1, k_2)\, C^{\dagger}_{k_2\uparrow} C^{\dagger}_{-k_2\downarrow} C_{-k_1\downarrow} C_{k_1\uparrow}$$

with $\xi_k = \varepsilon_k - \mu$.

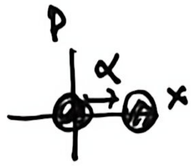
BCS - Ansatz

$$\psi_{BCS}(r_1 \cdots r_N ; \sigma_1 \cdots \sigma_N) = \mathcal{A}\{\varphi(r_1 r_2 ; \sigma_1 \sigma_2) \cdots \varphi(r_{N-1} r_N ; \sigma_{N-1} \sigma_N)\}$$

$$\varphi(r_1 r_2 ; \sigma_1 \sigma_2) = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2)\, \varphi(|r_1 - r_2|)$$

$$= \frac{1}{\sqrt{2}} \sum_k \chi(k)\, e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} (|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2)$$

$$= \sum_k \chi(k)\, C^{\dagger}_{k\uparrow} C^{\dagger}_{-k\downarrow} |vac\rangle$$



$$\rightarrow \psi_N = N^{-1/2} \left(\chi(k)\, C^{\dagger}_{k\uparrow} C^{\dagger}_{-k\downarrow}\right)^{N/2} |vac\rangle$$

displacement $|\alpha\rangle = N_\alpha e^{\alpha a^{\dagger}} |0\rangle$

$\xrightarrow{}$ releasing fixing N $\quad |\psi\rangle = \exp\left(\sum_k \chi(k)\, C^{\dagger}_{k\uparrow} C^{\dagger}_{-k\downarrow}\right) |vac\rangle$

$$= \prod_k \exp(\chi(k)\, C^{\dagger}_{k\uparrow} C^{\dagger}_{-k\downarrow}) |vac\rangle$$

$$= \prod_k (1 + \chi(k)\, C^{\dagger}_{k\uparrow} C^{\dagger}_{-k\downarrow}) |vac\rangle$$

$$\rightarrow \prod_k (u_k + v_k C^{\dagger}_{k\uparrow} C^{\dagger}_{-k\downarrow}) |vac\rangle$$

$|\psi\rangle$ is no-longer a slater determinant state, but a product of pairing ~~state~~ sharing the same phase of $v_k = |v_k| e^{i\phi}$

Consider the sub-Hilbert space $k\uparrow$ and $-k\downarrow$. They span 4 states $|0\rangle$, $c^\dagger_{-k\downarrow}|0\rangle$, $c^\dagger_{k\uparrow}|0\rangle$, $c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}|0\rangle$

even fermion number

$$\boxed{\begin{array}{ll} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}|0\rangle & 2(\varepsilon_k - \mu) \\ c^\dagger_{k\uparrow}|0\rangle \quad c^\dagger_{-k\downarrow}|0\rangle & : \ \varepsilon_k - \mu \\ |0\rangle & 0 \end{array}}$$

$$H_\Delta(k) = \Delta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \Delta_k^* c_{-k\downarrow} c_{k\uparrow}$$

basis $\leftarrow \begin{pmatrix} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}|0\rangle \\ |0\rangle \end{pmatrix}$

$$\Rightarrow H_\Delta(k) = \varepsilon_k - \mu + (\varepsilon_k - \mu)\cdot\tau_3 + \begin{bmatrix} 0 & \Delta_k \\ \Delta_k^* & 0 \end{bmatrix}$$

$$= \varepsilon_k - \mu + \underbrace{(\varepsilon_k - \mu)\tau_3 + \mathrm{Re}\,\Delta_k \tau_1 + \mathrm{Im}\,\Delta_k \tau_2}_{\text{Anderson's pesudo- spin}}$$

$$h_k = (\mathrm{Re}\,\Delta_k, \ \mathrm{Im}\,\Delta_k, \ \varepsilon_k - \mu)$$

excited pair

$$|EP\rangle = \alpha^\dagger_{k\uparrow} \alpha^\dagger_{-k\downarrow} |\Omega\rangle = (-v_k^* + u_k^* c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0\rangle$$

$\Rightarrow$

$c^\dagger_{k\uparrow}|0\rangle$ —— $\rightarrow$ $c^\dagger_{k\uparrow}|0\rangle$ $c^\dagger_{-k\downarrow}|0\rangle$ $\Big\} \sqrt{\xi_k^2 + \Delta^2}$

$|0\rangle$ —— $\rightarrow$ $\alpha^\dagger_{k\uparrow} \nwarrow \quad \nearrow \alpha^\dagger_{-k\downarrow} \Big\} \sqrt{\xi_k^2 + \Delta^2}$

$|\Omega\rangle \ (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0\rangle$

- Design operators for quasi-particle excitation (Bogoliubov)

$$\alpha^\dagger_{k\uparrow} = u_k^* c^\dagger_{k\uparrow} - v_k^* c_{-k\downarrow} \rightarrow \alpha_{k\uparrow} = u_k c_{k\uparrow} - v_k c_{-k\downarrow}$$

$$\alpha^\dagger_{-k\downarrow} = v_k^* c_{k\uparrow} + u_k^* c^\dagger_{-k\downarrow} \qquad \alpha_{-k\downarrow} = v_k c^\dagger_{k\uparrow} + u_k c_{-k\downarrow}$$

i.e.
$$\begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^\dagger_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^* & u_k^* \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} \Rightarrow \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k^* & v_k \\ -v_k^* & u_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha^\dagger_{-k\downarrow} \end{pmatrix}$$

$$\alpha^\dagger_{k\uparrow} \alpha^\dagger_{-k\downarrow} (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) |0\rangle = \alpha^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |0\rangle$$

$$= (-v_k^* + u_k^* c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) |0\rangle = |\,\mathrm{EP}\rangle$$

- Optimization

$$E = \langle K \rangle + \langle H_{int} \rangle$$

$$\langle K \rangle = \Big\langle \sum_{k\sigma} \xi_i c^\dagger_{k\sigma} c_{k\sigma} \Big\rangle = 2 \sum_k \xi_k |v_k|^2$$

$$\langle H_{int} \rangle = -\frac{1}{V} \sum_{k_1 k_2} g(k_1, k_2) \langle c^\dagger_{k_2\uparrow} c^\dagger_{-k_2\downarrow} \rangle \langle c_{-k_1\downarrow} c_{k_1\uparrow} \rangle$$

~~Define $F_k = u_k^* v_k = \langle 0 | c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow} | 1_{k\uparrow} 1_{-k\downarrow} \rangle$~~

$$\cancel{\langle H_{int} \rangle}\ E = 2 \sum_k \xi_k |v_k|^2 - \frac{1}{V} \sum_{k_1 k_2} g(k_1, k_2) u_{k_1}^* v_{k_1} u_{k_2} v_{k_2}^*$$

set $u_k = \cos\theta_k$, $v_k = \sin\theta_k$

$$E = 2 \sum_k \xi_k \sin^2\theta_k - \frac{1}{4} \sum_{k_1} \sin 2\theta_{k_1} \sum_{k_2} \frac{1}{V} g(k_1, k_2) \sin 2\theta_{k_2}$$

$$0 = \frac{\partial}{\partial \theta_k} E_k = 2\xi_k \sin 2\theta_k - \cos 2\theta_k \frac{1}{V} \sum_{k'} g(k k') \sin 2\theta_{k'}$$

Gap function: $\Delta_k = \frac{1}{2V} \sum_{k'} g(k k') \sin 2\theta_{k'}$

$$\Rightarrow \tan 2\theta_k = \frac{\Delta_k}{\xi_k} \Rightarrow \cos 2\theta_k = \frac{\xi_k}{E_k} \quad \sin 2\theta_k = \frac{\Delta_k}{E_k}$$

$$\boxed{\Delta_k = \frac{1}{V} \sum_{k'} g(k,k') \frac{\Delta_{k'}}{2E_{k'}} = \int \frac{dk'}{(2\pi)^3} g(k,k') \frac{\Delta_{k'}/2}{\sqrt{\xi_{k'}^2 + \Delta_{k'}^2}}}$$

finite temperature $\quad F = E - TS \quad S = -k_B \sum f_k \ln(1-f_k)$

or $\quad \Delta_k = \frac{1}{2V} \sum_{k'} g(k k') \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$

$$c_{-k'\downarrow} c_{k'\uparrow} = (-v_k \alpha^\dagger_{k\uparrow} + u_k^* \alpha_{-k\downarrow})(u_k^* \alpha_{k\uparrow} + v_k \alpha^\dagger_{-k\downarrow})$$

$$\rightarrow v_k u_k^* \left(-\langle \alpha^\dagger_{k\uparrow} \alpha_{k\uparrow} \rangle + \langle \alpha_{-k\downarrow} \alpha^\dagger_{-k\downarrow} \rangle\right)$$

$$= \sin\theta_k \cos\theta_k \left(1 - 2\frac{1}{e^{\beta E}+1}\right) = \frac{\sin 2\theta_k}{2} \tanh \frac{\beta E_k}{2}$$

$$\Rightarrow \boxed{\Delta_k = \int \frac{dk'}{(2\pi)^3} g(k,k') \frac{\Delta_{k'}/2}{E_{k'}} \tanh \frac{\beta E_{k'}}{2}}$$

BCS gap function!

4 Solution to the gap equation

Let us consider the simplest case that $|g(k,k')| = g$, i.e., Δ_k is independent on k, then Eq. 37 is simplified to

$$\Delta = gN_0 \int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon \frac{\Delta}{2E_k} \tanh\frac{\beta E_k}{2} \Rightarrow \frac{1}{gN_0} = \int_0^{\hbar\omega_D} d\epsilon \frac{1}{E_k} \tanh\frac{\beta E_k}{2}. \tag{42}$$

4.1 Gap function at zero temperature

First, consider the zero temperature, then by setting $\beta \to +\infty$ we have

$$\begin{aligned} \frac{1}{gN_0} &= \int_0^{\hbar\omega_D} d\epsilon \frac{1}{\sqrt{\epsilon^2+\Delta^2}} = \int_0^{\frac{\hbar\omega_D}{\Delta}} \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}\frac{\hbar\omega_D}{\Delta} \\ \Delta &= \frac{\hbar\omega_D}{\sinh\frac{1}{N_0 g}} \Rightarrow \Delta \approx 2\hbar\omega_D e^{-\frac{1}{N_0 g}} \quad \text{at} \quad gN_0 \ll 1. \end{aligned} \tag{43}$$

4.2 Gap function around T_c

Around T_c, $\Delta \to 0$, we have

$$\frac{1}{gN_0} = \int_0^{\hbar\omega_D} d\epsilon \frac{\tanh\frac{\beta_c\epsilon}{2}}{\epsilon} = \int_0^{\frac{1}{2}\beta_c\hbar\omega_D} dx \frac{\tanh x}{x}. \tag{44}$$

Since $\tanh x/x \to 1$ at small values of x, and $\to 1/x$ at large values of x, we can approximate $\int_0^a dx \frac{\tanh x}{x} \approx \int_0^1 dx + \int_1^a dx/x = \ln a + 1$ at $a \gg 1$. A more accurate estimate shows that

$$\int_0^a dx \frac{\tanh x}{x} \approx \ln 2.28x. \tag{45}$$

Hence, we have

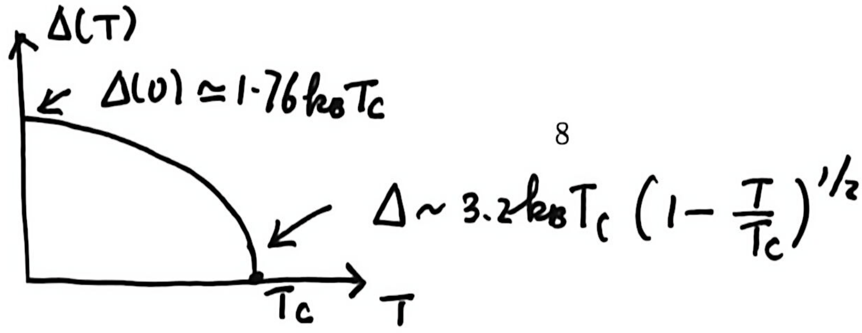
$$\frac{1}{gN_0} = \ln 1.14\beta_c\hbar\omega_D \Rightarrow k_B T_c = 1.14\hbar\omega_D e^{-\frac{1}{gN_0}} \tag{46}$$

A more careful analysis shows that the gap function $\Delta(T)$ at $T \to T_c$ behaves

$$\frac{\Delta}{k_B T_c} \sim 3.2\left(1 - \frac{T}{T_c}\right)^{\frac{1}{2}} \tag{47}$$

The sketch of the temperature dependence of the gap function $\Delta(T)$ based on the BCS theory is presented in Fig 2.

4.3 Universal relation between Δ and T_c



Compare Eq. 43 and Eq. 46, we have the universal relation for the weak coupling BCS superconductors that

Figure 2: The temperature dependence of the gap function $\Delta(T)$.

$$\frac{\Delta(T=0)}{k_B T_c} \approx 1.76. \tag{48}$$

This is a famous result of BCS theory, and widely used in literature for judge weak or strong coupling superconductivity. For strong coupling superconductors, it has significant deviations: $\frac{\Delta(T=0)}{k_B T_c} \approx 2.3$ and 2.6 for Hg and Pb, respectively. Certainly for high T_c superconductors, it can reach much larger values at the order of 10.

5 The isotope effect

The superconducting transition temperature $T_c = 1.14\hbar\omega_D e^{-\frac{1}{N_0 g}}$. ω_D inversely depends on $M_{ion}^{-\frac{1}{2}}$ due to Newton's equation, hence, $T_c \propto M^{-\frac{1}{2}}$. Historically, this isotope effect has been seen in a variety of superconductors such as Hg, Pb, Mg, Sn, Tl, etc. It was the motivation of electron-phonon mechanism for superconductivity.

6 The McMillan formula

So far, we have neglected the retarded nature of the phonon renormalized effective electron-electron interaction. In fact, such an interaction contains two parts: a screened Coulomb interaction, and a phonon-intermediated interaction. The overall effect is that the interaction is repulsive at $\omega > \omega_D$ and attractive at $\omega < \omega_D$. Let us neglect the angular dependence of $g_{k,k'}$ on $\mathbf{k} \cdot \mathbf{k}'$, but only keep their dependence on the radial part of momentum, which is equivalent to the energy dependence

$$-g(k,k') = V_c - V_{ph}(\frac{\xi_k - \xi_{k'}}{\hbar}). \tag{49}$$

This not a fully treatment to the retarded nature of the frequency dependent phonon-renormalized interaction. Rather, we approximate the interaction by its value at the on-shell frequency, but it is still constant of frequency. Then the gap equation close to T_c is reformatted as

$$\Delta(\xi) = -N_0 \int_{-\Lambda}^{\Lambda} d\xi' \Big(V_c - V_{ph}(\frac{\xi - \xi'}{\hbar})\Big)\Delta(\xi')\frac{\tanh\frac{\beta_c \xi'}{2}}{2\xi'}, \tag{50}$$

where $\Lambda \gg \hbar\omega_D$ is a high energy cut off. Basically, $\Delta(\xi)$ represents the gap function from those electrons with energy ξ to the Fermi surface.

We denote

$$A = -N_0 \int_{-\Lambda}^{\Lambda} d\xi' V_c \Delta(\xi')\frac{\tanh\frac{\beta_c \xi'}{2}}{2\xi'}, \tag{51}$$

then $\Delta(\xi) \approx A$ at $|\xi| \geq \hbar\omega_D$, where $V_{ph}(\omega)$ is already suppressed, and $\Delta(\xi)$ is nearly ξ-independent. Further, we define the average value of Δ over the region of $|\xi| \leq \hbar\omega_D$ is B. Then we have

$$B = N_0\bar{V}_{ph}\int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi' B\frac{\tanh\frac{\beta_c\xi'}{2}}{2\xi'} + A \approx BN_0\bar{V}_{ph}\ln\frac{\hbar\omega_D}{k_BT_c} + A, \tag{52}$$

where $\bar{V}_{ph}$ is the average of $V_{ph}(\xi-\xi')$ in the low frequency region.

On the other hand,

$$\begin{aligned} A &= -N_0\int_{-\Lambda}^{\Lambda} d\xi' V_c\Delta(\xi')\frac{\tanh\frac{\beta_c\xi'}{2}}{2\xi'} \\ &= -N_0V_c\Big(B\int_{-\hbar\omega_D}^{\hbar\omega_D} + A\int_{\hbar\omega_D}^{\Lambda} + A\int_{-\Lambda}^{-\hbar\omega_D}\Big)d\xi'\frac{\tanh\frac{\beta_c\xi'}{2}}{2\xi'} \\ &= -N_0V_c\Big(B\ln\frac{\hbar\omega_D}{k_BT_c} + A\ln\frac{\Lambda}{\omega_D}\Big). \end{aligned} \tag{53}$$

Combine the above two equations, we have the following relation

$$\begin{aligned} B\Big(1 - N_0\bar{V}_{ph}\ln\frac{\hbar\omega_D}{k_BT_c}\Big) &= A \\ A(1 + N_0V_c\ln\frac{\hbar\omega_c}{\Lambda}) &= -N_0V_c\ln\frac{\hbar\omega_D}{k_BT_c}B, \end{aligned} \tag{54}$$

hence,

$$\begin{aligned} 1 - N_0\bar{V}_{ph}\ln\frac{\hbar\omega_D}{k_BT_c} &= -\frac{N_0V_c\ln\frac{\hbar\omega_D}{k_BT_c}}{1+N_0V_c\ln\frac{\hbar\omega_c}{\Lambda}} \\ 1 = N_0\ln\frac{\hbar\omega_D}{k_BT_c}\Big(\bar{V}_{ph} &- \frac{V_c}{1+N_0V_c\ln\frac{\hbar\omega_c}{\Lambda}}\Big). \end{aligned} \tag{55}$$

Hence, we arrive at the famous McMillan formula for T_c

$$T_c = \hbar\omega_D\exp\Big\{-\frac{1}{N_0\bar{V}_{ph} - \mu^*}\Big\} \tag{56}$$

where μ^* reads

$$\mu^* = \frac{N_0V_c}{1+N_0V_c\ln\frac{\Lambda}{\omega_D}}. \tag{57}$$

μ^* reflect the renormalized Coulomb interaction to the superconductivity. It weakens the electron-phonon interaction strength from $N_0\bar{V}_ph \to N_0\bar{V}_ph - \mu^*$. A few remarks

1. Hence, Coulomb interaction is not sufficient to suppress superconductivity, due to the renormalized effect in the denominator.

2. The isotope effect is weakened due to the dependence of μ^* on ω_D. Enhancing ion mass lowers the Debye frequency ω_D, which certainly lowers T_c. On the other hand, the two energy scales of ω_D and Λ of phonons and Coulomb interaction are more separated, then the Coulomb interaction is more strongly renormalized. Hence μ^* is weakened which compensates partly the effect of lowering ω_D.

(*) Phase fluctuation T_θ v.s. Δ

binding energy of Cooper pair Δ

phase coherence energy scale $f = \frac{\hbar^2}{2m^*}|\psi|^2(\nabla\theta)^2$

$\nabla\theta \sim$ phase winding 2θ over distance ξ

$$k_B T_\theta \sim \xi^3 \frac{\hbar^2}{m^*}|\psi|^2\left(\frac{2\pi}{\xi}\right)^2 = \frac{4\pi^2\hbar^2}{m^*}\xi|\psi|^2$$

$$|\psi|^2 \sim \rho_s = \frac{|\alpha|}{\beta}$$

$$\lambda^2(T) = \frac{m^* c^2}{4\pi e^{*2}}\frac{\beta}{|\alpha|} \Rightarrow \rho_s = \frac{m^* c^2}{4\pi e^{*2}}\lambda^{-2}$$

$$k_B T_\theta \sim \frac{4\pi^2\hbar^2}{4\pi e^{*2}}c^2\xi\lambda^{-2} \sim \frac{1}{\alpha}\frac{hc}{\xi}\left(\frac{\xi}{\lambda}\right)^2$$

$$\xi \simeq \frac{\hbar v_F}{\Delta}$$

$$\boxed{\frac{k_B T_\theta}{\Delta} \sim \frac{1}{\alpha}\frac{c}{v_F}\left(\frac{\xi}{\lambda}\right)^2}$$

If $k_B T_\theta \gg \Delta$, then $k_B T_c \sim \frac{1}{1.76}\Delta$

but $k_B T_\theta \ll \Delta$, then $k_B T_c \sim k_B T_\theta$

	T_c(K)	T_θ/T_c
Pb	7 K	2×10^5
K_3C_{60}	19 K	17
YBCO	92 K	1.4