# Lecture 2 - d-wave superconductivity

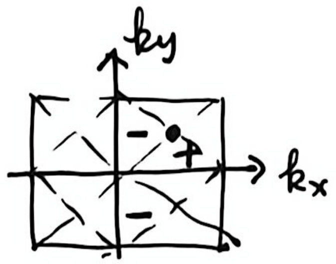
① Definition of unconventional symmetry

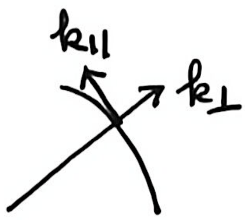
② Mean-field theory of d-wave SC

③ Thermodynamic properties (specific heat,
Knight shift)

④ superfluid density / optical conductivity

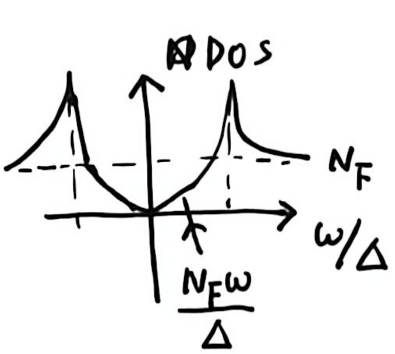
⑤ phase-sensitive : Corner Josephson junction.



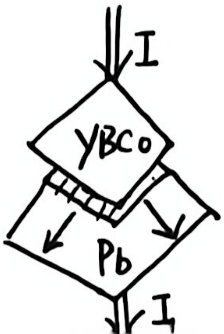

$$H(k) = \hbar v_F \delta k_{\perp} \tau_z + \Delta_d \, \delta k_{\parallel} \tau_x$$

$$\Delta_d = \Delta(\cos k_x - \cos k_y)$$

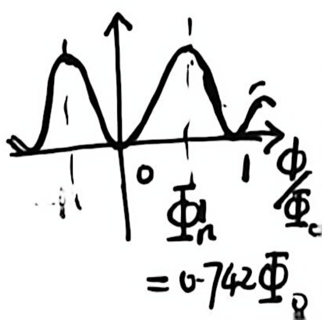



$$\lambda(T)/\lambda(0) \simeq 1 + \ln 2 \frac{k_B T}{\Delta}$$

$$\mathrm{Re}\,\sigma(\omega) / \sigma_n(\omega) \propto \omega^2$$




$$I_{max} = I_0 \frac{\sin^2 \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0}$$



# Unconventional superconductivity — a general view, d-wave pairing etc.

## § Definition: (real space picture)

The Cooper pairing structure can be classified by its symmetry property. Let us consider a strong coupling limit such that Cooper pairs can be viewed as diatom molecule whose real space wavefunctions can be written as

$$\psi_{\alpha_1\alpha_2}(\vec{r}_1, \vec{r}_2) = \Phi(\vec{R})\, \phi(\vec{r}_1 - \vec{r}_2)\, \chi_{\alpha_1\alpha_2}.$$

where $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$ is the center of mass coordinate, $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative coordinate, $\chi_{\alpha_1\alpha_2}$ is the spin-wave function. For simplicity, we assume $\Phi(\vec{R}) =$ constant, i.e. momentum zero pairing. In isotropic system, we can expand $\phi(\vec{r}_1 - \vec{r}_2)$ in terms of angular momentum basis. If no spin-orbit coupling. $\chi_{\alpha_1\alpha_2}$ can be classified as $\chi_s = \frac{|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle}{\sqrt{2}}$ (spin singlet)

and $\chi_{t,\, S_z = 1,0,-1} = \begin{cases} |\uparrow_1\rangle|\uparrow_2\rangle, \\ \frac{|\uparrow_1\rangle|\downarrow_2\rangle + |\downarrow_1\rangle|\uparrow_2\rangle}{\sqrt{2}}, \\ |\downarrow\rangle_1 |\downarrow_2\rangle \end{cases}$ (spin triplet).

Considering the fermionic statistics, $\psi_{\alpha_1\alpha_2}(\vec{r}_1, \vec{r}_2) = -\psi_{\alpha_2\alpha_1}(\vec{r}_2, \vec{r}_1)$, we have

$$\psi_{\alpha_1\alpha_2}(\vec{r}_1, \vec{r}_2) = \begin{cases} R_n(r)\, Y_{lm}(\vec{r})\, \chi_s & (\text{for } l = \text{even}) \\ R_n(r)\, Y_{lm}(\vec{r})\, \chi_{t,\,0,\pm 1} & (\text{for } l = \text{odd}). \end{cases}$$

* $R_n(r)$ is the radial wavefunction, and $n$ is the radial quantum number.

classification: according to symmetry.

① conventional pairing: S-wave, spin singlet. $R_{n=0}(r)$ positive definite. — Hg, Al, Pb, etc

② unconventional pairing: all other pairing symmetries except the s-wave.

example: d-wave high $T_c$ cuprates, singlet (Nobel prize)

p-wave $^3He$ - A and B phases, spin triplet (Nobel prizes)

$Sr_2RuO_4$ (?) almost

f-wave ? $UPt_3$

They may be nodal or nodeless, may be topologically trivial or not.

③ Extended s-wave: paring wavefunction does not change sign as varying angular variables, but changes sign along radial direction. (e.g. Iron-based superconductors, but not fully settled yet!)

(*) Unconventional pairing can save Coulumb repulsion energy since $\phi(\vec{r}=0)=0$. The propalbility of two electrons coincide at the same point vanishes!

§ Weak coupling (momentum space picture) — gap equation

TOPS
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cont.

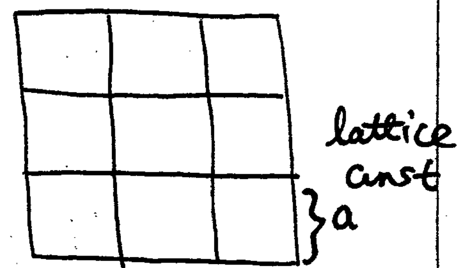
we first consider the unconventional pairing in the singlet channel.

- The simplest and most celebrate example is the high $T_c$ cuprates, whose physics mainly occurs in the 2D CuO plane. The lattice ~~structure~~ is square, and the rotation symmetry is only 4-fold.

Background:

The kinetic energy : tight-binding model

$$H_0 = -t \sum_{\langle ij\rangle} C_{i\sigma}^{+} C_{j\sigma} + h.c.$$




plug in Fourier component

$$C_i = \frac{1}{\sqrt{N}} \sum_{k} e^{i\vec{k}\cdot\vec{r}_i} C_k$$

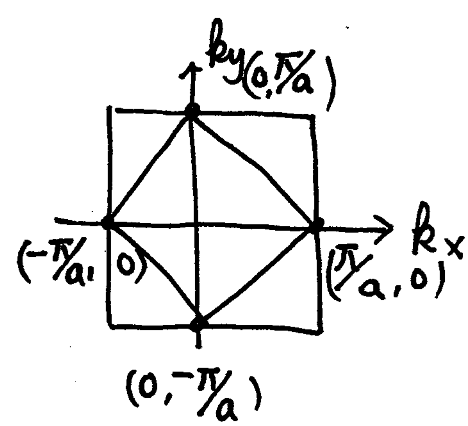
N is the number of lattice sites

$$\Rightarrow H_0 = \sum_{k} \varepsilon_k C_{k\sigma}^{+} C_{k\sigma} \quad \text{with } \varepsilon_k = -2t(\cos k_x + \cos k_y) - \mu$$

Ex: ① please derive the $H_0$ in momentum space.

② prove that at half filling, i.e. $\langle n\rangle = \langle C_{i\sigma}^{+} C_{i\sigma}\rangle = 1$.
The chemical potential $\mu = 0$, and the Fermi surface has the shape of a diamond. i.e.

$$\boxed{\cos k_x + \cos k_y = 0}$$




③ please also plot Fermi surfaces for negative values of $\mu$, say $|\mu/t| = 0.05, 0.1$, etc.
This corresponds to the situation of doping.

Due to the strong onsite Coulomb interaction, we consider the pairing on NN bonds. (The mechanism for the gluing force remains unknown).

$$H_{int} = -\frac{V}{2} \sum_{\delta = \pm\hat{x}, \pm\hat{y}} \left( C^\dagger_{i+\delta\downarrow} C^\dagger_{i\uparrow} - C^\dagger_{i+\delta\uparrow} C^\dagger_{i\downarrow} \right) \left( C_{i\uparrow} C_{i+\delta\downarrow} - C_{i\downarrow} C_{i+\delta\uparrow} \right)$$

— phenomenological interaction leading to d-wave pairing

perform Fourier transformation, and keep the pairing term

$$H_{pair} = -\frac{V}{2N} \sum_{k',k} \sum_{\vec{\delta}} e^{i\vec{k}'\cdot\vec{\delta}} \cdot e^{-i\vec{k}\cdot\vec{\delta}} \left[ C^\dagger_{-k'\downarrow} C^\dagger_{k'\uparrow} - C^\dagger_{-k'\uparrow} C^\dagger_{k'\downarrow} \right] \left[ C_{k\uparrow} C_{-k\downarrow} - C_{k\downarrow} C_{-k\uparrow} \right]$$

$$= -\frac{V}{2N} \sum_{k'k} 4 \left( \cos k'_x \cos k_x + \cos k'_y \cos k_y \right) C^\dagger_{-k'\downarrow} C^\dagger_{k'\uparrow} C_{k\uparrow} C_{-k\downarrow}$$

$$= -\frac{V}{N} \sum_{k'k} \Big\{ \left( \cos k'_x + \cos k'_y \right) C^\dagger_{-k'\downarrow} C^\dagger_{k'\uparrow} \left( \cos k_x + \cos k_y \right) C_{k\uparrow} C_{-k\downarrow}$$

$$+ \left( \cos k'_x - \cos k'_y \right) C^\dagger_{-k'\downarrow} C^\dagger_{k'\uparrow} \left( \cos k_x - \cos k_y \right) C_{k\uparrow} C_{-k\downarrow} \Big\}$$

Define $\Delta_s = \frac{V}{N} \sum_k C_{k\uparrow} C_{-k\downarrow} \left( \cos k_x + \cos k_y \right)$

$$\Delta_d = \frac{V}{N} \sum_k C_{k\uparrow} C_{-k\downarrow} \left( \cos k_x - \cos k_y \right)$$

$$\frac{1}{N} H_{MF} = -\frac{V}{N} \sum_k \Delta^*_s \left( \cos k_x + \cos k_y \right) C_{k\uparrow} C_{-k\downarrow} + h.c$$

$$- \frac{V}{N} \sum_k \Delta^*_d \left( \cos k_x - \cos k_y \right) C_{k\uparrow} C_{-k\downarrow} + h.c$$

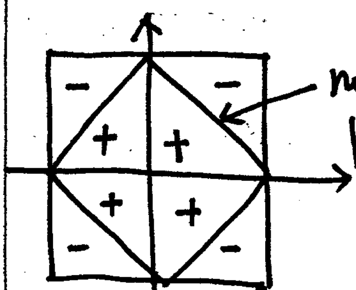
$$+ \frac{1}{V} \left( \Delta^*_s \Delta_s + \Delta^*_d \Delta_d \right)$$

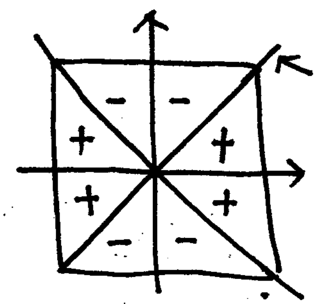
We have chosen the interaction $V(kk') = V_0(\cos k_x' \cos k_x + \cos k_y' \cos k_y)$.

This interaction can give rise to two possible singlet pairing symmetries:

the extended s-wave : gap function $\Delta_s(\cos k_x + \cos k_y)$

d-wave: gap function $\Delta_d(\cos k_x - \cos k_y)$.



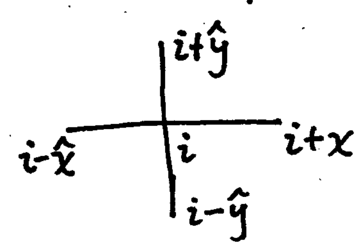

rotational invariant
but changes sign acrossing
$|k_x|+|k_y|=\pi$.

rotate $90^\circ$
$\Delta d \to -\Delta d$

Ex: please perform Fourier transformation back to real space.

① $\Delta_s$ corresponds to the real space pattern

$$\Delta(i,i+x) = \Delta(i,i-x) = \Delta(i,i+\hat{y}) = \Delta(i,i-\hat{y})$$




② $\Delta_d$ corresponds to the pattern

$$\Delta(i,i+x) = \Delta(i\ i-x) = -\Delta(i,i+\hat{y}) = -\Delta(i,i-\hat{y})$$

where $\Delta(i,i+\delta) = V\langle C_{i\uparrow} C_{i+\delta\downarrow} - C_{i\downarrow} C_{i+\delta\uparrow}\rangle$.

The extended s-wave and d-wave compete, and the d-wave pairing wins. The reason is that the nodal lines of $\Delta_s$ coincide with the Fermi surface at half-filling (For high Tc cuprates, the filling is very close to half-filling), thus the gap function is suppressed on Fermi surface. Now let us only keep the d-wave channal.

$$\frac{H}{N} = \frac{1}{N}\sum_k \begin{pmatrix} C_{k\uparrow}^{+} & C_{-k\downarrow} \end{pmatrix} \begin{pmatrix} \varepsilon_k-\mu & \Delta_d(\cos k_x-\cos k_y) \\ \Delta_d^{*}(\cos k_x - \cos k_y) & -(\varepsilon_k-\mu) \end{pmatrix} \begin{pmatrix} C_{k\uparrow} \\ C_{-k\downarrow}^{+} \end{pmatrix}$$

$$+ \frac{1}{N}\sum_k (\varepsilon_k-\mu) + \frac{1}{V}\Delta_d^{*}\Delta_d$$

Introducing Bogoliubov transformation, and assum $\Delta_d$ is real

$$\begin{pmatrix} C_{k\uparrow} \\ C_{-k\downarrow}^{+} \end{pmatrix} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \beta_{-k\downarrow}^{+} \end{pmatrix}$$

we have

$$\rightarrow \begin{pmatrix} \alpha_{k\uparrow}^{+} & \beta_{-k\downarrow} \end{pmatrix} \underbrace{\begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} \xi_k & \Delta(k) \\ \Delta(k) & -\xi_k \end{pmatrix} \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}} \begin{pmatrix} \alpha_{k\uparrow} \\ \beta_{-k\downarrow}^{+} \end{pmatrix}$$

$$\Downarrow$$

$$= \begin{bmatrix} \xi_k\cos 2\theta_k + \Delta(k)\sin 2\theta_k, & -\xi_k\sin 2\theta_k + \Delta(k)\cos 2\theta_k \\ -\xi_k\sin 2\theta_k + \Delta(k)\cos 2\theta_k, & -\xi_k\cos 2\theta_k - \Delta(k)\sin 2\theta_k \end{bmatrix}$$

$$(\xi_k = \varepsilon_k - \mu, \quad \Delta(k) = \Delta_d(\cos k_x - \cos k_y))]$$

Set $\tan 2\theta_k = \dfrac{\Delta(k)}{\xi_k}$ $\quad \cos 2\theta_k = \dfrac{\xi_k}{E_k}$, $\sin 2\theta_k = \dfrac{\Delta_k}{E_k}$

with $E_k = \sqrt{\xi_k^2 + \Delta^2(k)}$

$$\Rightarrow \frac{H}{N} = \frac{1}{N}\sum_k E_k \left[ (\alpha_{k\uparrow}^+ \alpha_{k\uparrow} - 1/2) + (\beta_{-k\downarrow}^+ \beta_{-k\downarrow} - 1/2) \right]$$

$$+ \frac{1}{N}\sum_k (\varepsilon_k - \mu) + \frac{1}{V}|\Delta_d|^2,$$

$$\cos^2\theta_k = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right), \quad \sin^2\theta_k = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right).$$

§ Self-consistency

$$\frac{F}{N} = \frac{1}{N}\sum_k -\frac{2}{\beta}\ln\left(e^{\frac{\beta}{2}E_k} + e^{-\frac{\beta}{2}E_k}\right) + \frac{1}{N}\sum_k (\varepsilon_k - \mu) + \frac{\Delta_d^2}{V}$$

$$= -\frac{2}{\beta}\int_{FBZ}\frac{d^2k}{(2\pi)^2}\ln 2\cosh\frac{\beta}{2}E_k + \frac{\Delta_d^2}{V} + \text{const}$$

$$\frac{\partial F}{\partial \Delta_d} = -\frac{2}{\beta}\int_{FBZ}\frac{d^2k}{(2\pi)^2}\frac{\sinh\frac{\beta}{2}E_k}{\cosh\frac{\beta}{2}E_k}\cdot\frac{\beta}{2}\frac{\Delta_d(\cos k_x - \cos k_y)^2}{E_k} + \frac{2\Delta_d}{V} = 0$$

$$\Rightarrow \boxed{\Delta_d = V\int_{FBZ}\frac{d^2k}{(2\pi)^2}\frac{\Delta_d(\cos k_x - \cos k_y)^2}{2E_k}\tanh\frac{\beta}{2}E_k.}$$ ← Gap equation "d-wave".

$$n = -\frac{1}{N}\frac{\partial F}{\partial \mu} \Rightarrow \frac{1}{N}\frac{\partial F}{\partial \mu} = -\frac{2}{\beta}\int_{FBZ}\frac{d^2k}{(2\pi)^2}\tanh\frac{\beta}{2}E_k\cdot\frac{\beta}{2}\frac{-\xi_k}{E_k} - 1 = -n$$

$$\Rightarrow \boxed{1 - n = \int_{FBZ}\frac{d^2k}{(2\pi)^2}\tanh\frac{\beta}{2}E_k\cdot\frac{\xi_k}{E_k}}$$ ← particle number

Gap equation: cf. the general form of gap equation:

$$\boxed{\Delta(k) = \int\frac{d^dk'}{(2\pi)^d}V(k,k')\frac{\Delta(k')}{2\sqrt{\xi^2 + \Delta^2(k')}}\tanh\frac{\beta}{2}\sqrt{\xi^2 + \Delta^2(k')}}$$

plug in $\Delta(k) = \Delta_d(\cos k_x - \cos k_y)$, $V(k,k') = V(\cos k_x - \cos k_y)(\cos k_x' - \cos k_y')$.

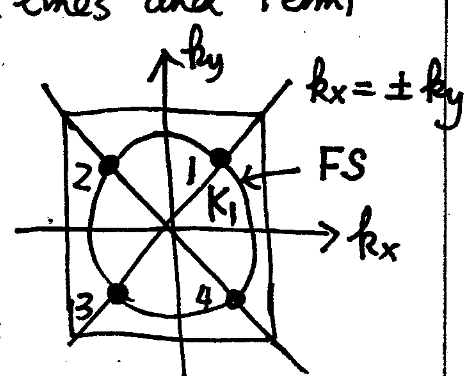
we will get the d-wave gap equation.

(*) Dirac spectra (nodal quasi-particle)

$$\pm E_k = \pm\sqrt{\xi_k^2 + \Delta^2(k)} \quad : \quad \xi_k = 0 \text{ (Fermi surface)}$$

$$\Delta(k) = 0 \quad \text{gap nodal line}$$

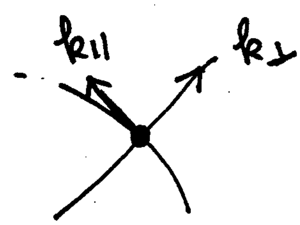
zeros of $E_k$: crossing points of gap nodal lines and Fermi surface. There are nodal points.




"d"-wave superconductivity is NOT fully gapped, but gapless. The nodal quasi-particles dominates over the low energy thermodynamic properties!

Let us linearize the d-wave Hamiltonian around one of the nodes, say, node 1.

$$\begin{cases} \xi_k = \hbar v_F \delta k_\perp \\ \Delta(k) = \Delta_d \delta k_{\parallel} a \end{cases}$$




where
$$\begin{cases} \delta k_\perp = \dfrac{\delta k_x + \delta k_y}{\sqrt{2}} \\ \delta k_{\parallel} = \dfrac{-\delta k_x + \delta k_y}{\sqrt{2}} \end{cases} \quad \text{and} \quad \delta\vec{k} = \vec{k} - \vec{K}_1$$

$$\boxed{H(k) = \begin{pmatrix} \xi_k & \Delta(k) \\ \Delta(k) & -\xi_k \end{pmatrix} = \hbar v_F \delta k_\perp \tau_z + \Delta_d \delta k_{\parallel} \tau_x}$$

# Thermodynamics of nodal superconductors (singlet)

2D-d-wave

In this lecture, we will study new features associated with the nodal quasi-particles of the d-wave superconductors. The d-wave gap equation can be solved analytically in the continuum approximation:

① Assum $\xi_k$ is isotropic, i.e. independent of the azimuthal angle $\varphi_k$

$$\int \frac{d^2k}{(2\pi)^2} \longrightarrow \int \frac{d\varphi}{2\pi} \int_{-\omega_0}^{\omega_0} d\xi \; \rho_0(\xi),$$ where $\rho_0(\xi)$ is the density of states. If $\rho_0(\xi)$ doesnot have sigularity, it can be replace by $N_F$, i.e. the DOS right at Femi surface. $\omega_0$ is the cut off, which plays the role of Debye frequency in conventional SC. In high $T_c$, the origin of $\omega_0$ is still in debate, most probably, it arises from antiferro-magnetic fluctuations.

② we replace the lattice version of the angular form factor $\cos k_x - \cos k_y$ by $\cos 2\varphi_k$, which has the same $d_{x^2-y^2}$ symmetry. An issue is the normalization, which can absorbed in the definition of $\Delta_d$ and $V$. Say, $\cos k_x - \cos k_y \sim C \cdot \cos 2\varphi_k$

$$\Rightarrow \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{\omega_0} d\xi \; \frac{C^2 \cos^2 2\varphi_k}{\sqrt{\xi^2 + \Delta_d^2 C^2 \cos^2 2\varphi_k}} \tanh \frac{\beta}{2} \sqrt{\xi^2 + \Delta_d^2 C^2 \cos^2 2\varphi_k} = \frac{1}{N_F V}$$

we can define $\frac{1}{VC^2} \rightarrow \frac{1}{V}$ and $\Delta_d^2 C^2 \rightarrow \Delta_d^2$, we have

$$\boxed{\int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{\omega_0} d\xi \; \frac{\cos^2 2\varphi}{\sqrt{\xi^2 + \Delta_d^2 \cos^2 2\varphi}} \tanh \frac{\beta}{2} \sqrt{\xi^2 + \Delta_d^2 \cos^2 \varphi} = \frac{1}{N_F V}}$$

* Solve $T_c$

we have around $T_c$, $\underbrace{}_{\Delta=0}$ $\Rightarrow$ $$\int_0^{\frac{\omega_0}{2k_BT_c}} dx\ \frac{\tanh x}{x} = \frac{2}{N_F V}$$

( $x = \frac{\beta_c}{2}\xi$, the "2" factor on RHS comes from $\int \frac{d\varphi}{2\pi}\gamma_\varphi^2 = \frac{1}{2}$ ).

Integral by part $\Rightarrow$ LHS= $\ln\frac{\omega_0}{2k_BT_c}\tanh\frac{\omega_0}{2k_BT_c} - \int_0^{\frac{\omega_0}{2k_BT_c}} dx\ \ln x\ \mathrm{sech}^2 x$

we can set $\frac{\omega_0}{2k_BT_c} \to \infty$ converge!

define $C_0 = \frac{1}{2}\exp\left[-\int_0^\infty dx\ \frac{\ln x}{\cosh^2 x}\right]$

$= 1.134$

$$\Rightarrow \boxed{k_B T_c \approx C_0\,\omega_0\, e^{-\frac{2}{N_F V}}}$$

Because $\omega_0$, $V$ are difficult to know, this equation does not tell much useful information!

* Solve gap value at $T=0$,

$\beta\to\infty$, $\tanh\frac{\beta}{2}E \to 1$. we have

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\int_0^{\omega_0} d\xi\ \frac{\cos^2 2\varphi}{\sqrt{\xi^2+\Delta_d^2\cos^2 2\varphi}} = \frac{1}{N_F V}$$

$$\int dx\ \frac{1}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C$$

$$\Rightarrow \int_0^{\omega_0} d\xi\ \frac{1}{\sqrt{\xi^2+\Delta_d^2\cos^2 2\varphi}} = \ln\left(\xi+\sqrt{\xi^2+\Delta_d^2\cos^2 2\varphi}\right)\Big|_0^{\omega_0} = \ln\frac{\omega_0+\sqrt{\omega_0^2+\Delta_d^2\cos^2 2\varphi}}{\Delta_d|\cos 2\varphi|}$$

$$\Rightarrow \int_0^{\pi} d\varphi\ \cos^2 2\varphi\ \ln\frac{\omega_0+\sqrt{\omega_0^2+\Delta_d^2\cos^2 2\varphi}}{\Delta_d|\cos 2\varphi|} = \frac{\pi}{N_F V}$$

Consider the case of $\omega_0 \gg \Delta_d$, we can approximate the integral as

$$\int_0^{\pi} d\varphi \cos^2 2\varphi \ln \frac{2\omega_0}{\Delta_d |\cos 2\varphi|} \simeq \frac{\pi}{N_F V}$$

$$\Rightarrow \frac{\pi}{2} \ln \frac{2\omega_0}{\Delta_d} = \frac{\pi}{N_F V} + \int_0^{\pi} d\varphi \cos^2 2\varphi \ln|\cos 2\varphi|$$

$2\int_0^{\pi/2} d\varphi \cos^2\varphi \ln \cos\varphi$

$$\Rightarrow \frac{2\omega_0}{\Delta_d} = \frac{2}{N_F V} + \frac{4}{\pi}\int_0^{\pi/2} d\varphi \cos^2\varphi \ln\cos\varphi$$

$$\Rightarrow \boxed{\Delta_d = C_1 \omega_0 e^{-\frac{2}{N_F V}}}, \quad \text{where } C_1 = 2\cdot\exp\left[-\frac{4}{\pi}\int_0^{\pi/2} d\varphi \cos^2\varphi \ln\cos\varphi\right]$$

$\simeq 2.426$

we arrive the relation between gap and $T_c$.

$$\boxed{\frac{2\Delta_d}{k_B T_c} \simeq 4.28},$$

which is slightly higher than the s-wave value 3.53.

## § DOS in d-wave superconductor

$$\rho(\omega) = \frac{2}{Vol}\sum_k \left(u_k^2\,\delta(\omega - E_k) + v_k^2\,\delta(\omega + E_k)\right)$$

$$= 2\int \frac{d^2k}{(2\pi)^2}\,\frac{1}{2}\left[\left(1 + \frac{\xi_k}{E_k}\right)\delta(\omega - E_k) + \left(1 - \frac{\xi_k}{E_k}\right)\delta(\omega + E_k)\right]$$

$$= \int\frac{d\varphi}{2\pi}\int d\xi\,\frac{N_F}{2}\left[\left(1+\frac{\xi}{E}\right)\delta(\omega - E) + \left(1 - \frac{\xi}{E}\right)\delta(\omega + E)\right]$$

← odd function

consider $\omega > 0 \Rightarrow \rho(\omega) = \int\frac{d\varphi}{2\pi}\int d\xi\, N_F \left(1 + \frac{\xi}{E}\right)\delta(\omega - E)$

$$= \int\frac{d\varphi}{2\pi}\int_0 d\xi\,\frac{N_F}{2}\,\delta\left(\omega - \sqrt{\xi^2 + \Delta_d^2\cos^2 2\varphi}\right)$$

The solution of $\omega^2 = \xi^2 + \Delta_d^2 \cos^2 2\varphi \Rightarrow \xi = \pm\sqrt{\omega^2 - \Delta_d^2\cos^2 2\varphi}$

$$\Rightarrow \delta(\omega - E) = \frac{\delta\left(\xi - \sqrt{\omega^2 - \Delta^2\cos^2 2\varphi}\right) + \delta\left(\xi + \sqrt{\omega^2 - \Delta^2\cos^2 2\varphi}\right)}{|\xi| / \sqrt{\xi^2 + \Delta_d^2 \cos^2\varphi}}$$

$$\delta(\omega - E) = \frac{\omega}{\sqrt{\omega^2-\Delta^2\cos^2 2\varphi}}\left[\delta(\xi - \sqrt{\omega^2-\Delta^2\cos^2 2\varphi}) + \delta(\xi + \sqrt{\omega^2-\Delta^2\cos^2 2\varphi})\right]$$

$$\rho(\omega) = \frac{N_F}{2}\int\frac{d\varphi}{2\pi}\int d\xi \, \frac{\omega}{\sqrt{\omega^2-\Delta^2\cos^2 2\varphi}}\left[\delta(\xi - \sqrt{\omega^2-\Delta^2\cos^2 2\varphi}) + \delta(\xi + \sqrt{\omega^2-\Delta^2\cos^2 2\varphi})\right]$$

$$= N_F\int_0^{2\pi}\frac{d\varphi}{2\pi}\frac{\omega}{\sqrt{\omega^2-\Delta^2\cos^2 2\varphi}}\,\theta(\omega > |\Delta\cos 2\varphi|)$$

$$= N_F\,\frac{1}{2}\int^{4\pi}\frac{d\varphi'}{2\pi}\frac{\omega}{\sqrt{\omega^2-\Delta^2\cos^2\varphi'}}\,\theta(\omega>|\Delta\cos 2\varphi|) = \frac{2N_F}{\pi}\int_0^{\pi/2}d\varphi'\frac{\omega}{\sqrt{\omega^2-\Delta^2\cos^2\varphi'}}\,\theta(\cdots)$$

① if $\omega > \Delta \Rightarrow$ $\rho(\omega) = \frac{2N_F}{\pi}\int_0^{\pi/2}d\varphi'\frac{1}{\sqrt{1-(\Delta/\omega)^2\cos^2\varphi'}} = \frac{2N_F}{\pi}\int_0^{\pi/2}d\varphi'\frac{1}{\sqrt{1-\left(\frac{\Delta}{\omega}\right)^2\sin^2\varphi'}}$

This is the complete Elliptic integral of 1st kind.

as $\boxed{\begin{array}{ll}\Delta/\omega \to 1, \text{ we have} & \rho(\omega) \simeq \frac{N_F}{\pi}\ln\frac{8}{1-\Delta/\omega} \\ \Delta/\omega \to \infty & \rho(\omega) = N_F\end{array}}$

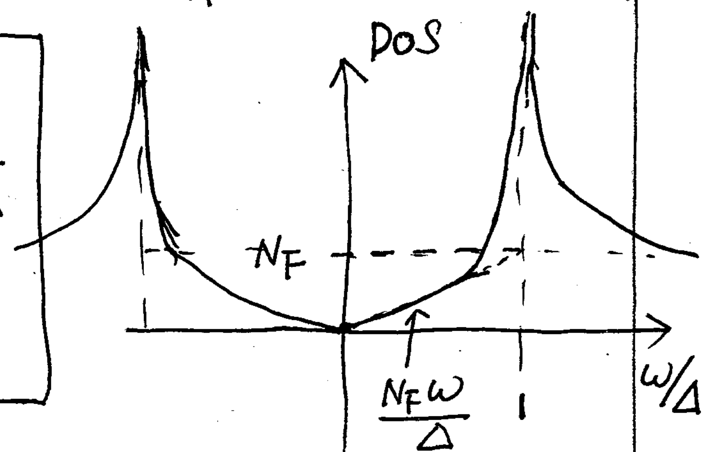
② if $\omega < \Delta$ define $\cos\phi = \frac{\Delta}{\omega}\cos\varphi'$ $\Rightarrow$ $\sin\phi\, d\phi = \frac{\Delta}{\omega}\sin\varphi'\, d\varphi'$

$$\rho(\omega) = \frac{2N_F}{\pi}\int_0^{\pi/2}\left(\frac{\Delta}{\omega}\right)^{-1}\frac{\sin\phi}{\sin\varphi'}\,d\phi\cdot\frac{1}{\sqrt{1-\cos^2\phi}}$$

$$\frac{\omega}{\Delta}\,\frac{2N_F}{\pi}\int_0^{\pi/2}\frac{d\phi}{\sin\varphi'} \leftarrow \sin\varphi' = \sqrt{1-\cos^2\varphi'} = \sqrt{1-\left(\frac{\omega}{\Delta}\right)^2\cos^2\phi}$$

$$= \frac{\omega}{\Delta}\,\frac{2N_F}{\pi}\int_0^{\pi/2}d\phi\,\frac{1}{\sqrt{1-\left(\frac{\omega}{\Delta}\right)^2\cos^2\phi}} = \frac{\omega}{\Delta}\,\frac{2N_F}{\pi}\int_0^{\pi/2}d\phi\,\frac{1}{\sqrt{1-\left(\frac{\omega}{\Delta}\right)^2\sin^2\phi}}$$

$$\boxed{\begin{array}{l}\text{as } \omega\to\Delta \quad \rho(\omega) \to \frac{N_F\,\omega}{\pi\,\Delta}\ln\frac{8}{1-\omega/\Delta} \\ \text{as } \omega\to 0 \quad \rho(\omega) \to \frac{N_F\,\omega}{\Delta}\end{array}}$$



§ Specific heat

$\leftarrow \text{Vol} = N a^2$

$a$: lattice constant

The free energy density $\frac{F(T)}{\text{Vol}} = -k_B T \ln Z$

$$\frac{F(T)}{\text{Vol}} = -k_B T \frac{1}{\text{Vol}} \sum_k 2 \ln\left(e^{-\frac{1}{2}\beta E_k} + e^{\frac{1}{2}\beta E_k}\right) + \frac{\Delta_d^2}{V}$$

$$= -k_B T \int_{FBZ} \frac{d^2k}{(2\pi)^2}\, 2 \ln 2\cosh \frac{\beta}{2} E_k + \frac{\Delta_d^2}{V}$$

$$\frac{S}{\text{Vol}} = -\frac{\partial F}{\text{Vol}\,\partial T} = k_B \int \frac{d^2k}{(2\pi)^2}\, 2\ln 2\cosh\frac{\beta}{2}E_k + k_B T \int \frac{d^2k}{(2\pi)^2}\, 2\tanh\frac{\beta}{2}E_k \frac{\partial}{\partial T}\left(\frac{\beta E_k}{2}\right)$$
$$- \frac{2\Delta_d}{V}\frac{\partial \Delta_d}{\partial T}$$

gap Eq: $\frac{\Delta_d}{V} = \int \frac{d^2k}{(2\pi)^2} \Delta_d \frac{(\cos k_x - \cos k_y)^2}{2E_k} \tanh\frac{\beta}{2}E_k$

$$\frac{\partial}{\partial T}\left(\frac{\beta}{2}E_k\right) = \frac{-1}{2k_B T^2} E_k + \frac{\beta}{2} \frac{\Delta_d (\cos k_x - \cos k_y)^2}{E_k} \frac{\partial \Delta}{\partial T}$$

$$\Rightarrow \frac{S}{\text{Vol}} = 2k_B \int \frac{d^2k}{(2\pi)^2} \ln\left(2\cosh\frac{\beta}{2}E_k\right) - 2k_B \int \frac{d^2k}{(2\pi)^2} \tanh\frac{\beta}{2}E_k \frac{\beta E_k}{2} \quad \text{(other term cancels)}$$

Ex: check $\frac{S}{\text{Vol}}$ can also be written as

$$\frac{S}{\text{Vol}} = -2k_B \sum_k \left[(1-f_k)\ln(1-f_k) + f_k \ln f_k\right]$$

with $f_k = \frac{1}{e^{\beta E_k}+1}$. Check it's consistent with the above Eq.

$$\frac{C}{\text{Vol}} = T\frac{\partial S}{\partial T} = -\beta \frac{\partial S}{\partial \beta} = 2\beta k_B \frac{1}{\text{Vol}} \sum_k \frac{\partial f_k}{\partial \beta} \ln \frac{f_k}{1-f_k} = -2\beta^2 k_B \frac{1}{\text{Vol}} \sum_k E_k \frac{\partial f_k}{\partial \beta}$$

$$= -2\beta^2 k_B \frac{1}{\text{Vol}} \sum_k E_k \frac{df_k}{d(\beta E_k)} \left(\frac{d(\beta E_k)}{d\beta}\right) \leftarrow \frac{d\beta E_k}{d\beta} = E_k + \beta \frac{dE_k}{d\beta}$$

$$= 2\beta k_B \frac{1}{\text{Vol}} \sum_k \left(-\frac{\partial f_k}{\partial E_k}\right)\left(E_k^2 + \frac{1}{2}\beta \frac{dE_k^2}{d\beta}\right)$$

$$\frac{C}{Vol} = 2k_B \int \frac{d^2k}{(2\pi)^2} \frac{e^{\beta E_K}}{(e^{\beta E_K}+1)^2}\left(\frac{E_K^2}{k_B^2T^2} - \frac{1}{2}\frac{dE_k^2}{k_B^2 T dT}\right)$$

① Next we consider low T limit.

Now we use continuum approx: $E_k^2 = \xi^2 + \Delta_d^2 \cos^2 2\varphi_k$

$$\frac{E_k^2}{k_B^2T^2} = \frac{\xi^2 + \Delta_d^2(T)\cos^2 2\varphi_K}{k_B^2 T^2}$$

$$\frac{E_K dE_K}{k_B^2\, TdT} = \frac{\Delta_d(T)}{k_BT}\frac{d\Delta_d(T)}{k_B dT}\cos^2 2\varphi_K = \frac{\Delta^2(T)}{(k_BT)^2}\cos^2 2\varphi_K\left(\frac{k_BT}{\Delta_d(T)}\right)^3$$

because at $T \ll \Delta(T)$, $\dfrac{d\Delta(T)}{k_B dT} \simeq \dfrac{k_B^2T^2}{\Delta_d^2(T)}$ (for d-wave),

we can neglect the contribution from the second term.

$\Rightarrow$ at $T \ll \Delta$,

$$\frac{C}{k_B} = \frac{1}{k_B^2T^2} N_F \int\frac{d\varphi}{2\pi}\int d\xi \frac{e^{\beta E}}{(e^{\beta E}+1)^2} E^2$$

$$= \frac{N_F}{4k_B^2T^2}\int\frac{d\varphi}{2\pi}\int_{-\infty}^{+\infty} d\xi \frac{E^2}{\cosh^2(E/2T)}$$

The factor $\cosh^2 E/2T$ suppresses the contribution except from the nodal region:

$$|\xi| > |\Delta||\cos 2\varphi| \Rightarrow \Delta\varphi \sim |\varphi - \pi/4| < \frac{|\xi|}{2|\Delta|}$$

consider there're four nodes,

$$\frac{C}{k_B} = \frac{N_F}{k_B^2T^2}\int_{-\infty}^{+\infty} d\xi \frac{\xi^2}{\cosh^2(\xi/2T)}\int_{-\frac{|\xi|}{2|\Delta|}}^{\frac{|\xi|}{2\Delta}} d\Delta\varphi + o(e^{-\Delta/T})$$

$$\approx \frac{N_F}{k_B^2T^2}\frac{1}{\Delta}\int_{-\infty}^{+\infty} d\xi \frac{|\xi|^3}{\cosh^2 \xi/2Tk_B}$$

defin $x = \dfrac{\xi}{2Tk_B}$

$$\frac{C}{k_B} \approx \frac{2^5 N_F k_B^2 T^2}{\Delta}\int_0^{+\infty} dx \frac{x^3}{\cosh^2 x} \simeq \text{const.}\frac{N_F (k_BT)^2}{\Delta}$$

The low ~~temperature~~ specific heat in 2D nodal SC

$$\frac{C}{k_B} \simeq \text{const.} \frac{N_F (k_B T)^2}{\Delta_d},$$ which is

Consistent with the low energy DOS $\simeq N_F \frac{\omega}{\Delta_d}$

Paramagnetic susceptibility / Knight shift

consider the pairing sector $k\uparrow$ and $-k\downarrow$. The Hilbert space is 4-dimensional: $|\Omega\rangle$, $\alpha^{+}_{k\uparrow}|\Omega\rangle$, $\beta^{+}_{-k\downarrow}|\Omega\rangle$ $\alpha^{+}_{k\uparrow}\beta^{+}_{-k\downarrow}|\Omega\rangle$.

The partition function $1+e^{-\beta E_k}+e^{-\beta E_k}+e^{-2\beta E_k}=(1+e^{-\beta E_k})^2$

if adding external field, $E_{\alpha k\uparrow} = E_k - \mu_B H$

$E_{\beta -k\downarrow} = E_k + \mu_B H$

$$\Rightarrow M = \mu_B \sum_k \frac{e^{-\beta(E_k-\mu_B H)}}{(1+e^{-\beta E_k})^2} - \frac{e^{-\beta(E_k+\mu_B H)}}{(1+e^{-\beta E_k})^2}$$

we neglect the dependence on H in the denominator at the linear order of H

$$\Rightarrow \chi = \frac{\partial M}{\partial H} = \mu_B^2 \sum_k \frac{e^{-\beta E_k}}{(1+e^{-\beta E_k})^2}(2\beta)$$

$$\Rightarrow \frac{\chi}{Vol} = \beta\mu_B^2 N_F \int d\xi \int \frac{d\varphi}{2\pi} \frac{1}{(e^{-\beta E/2}+e^{\beta E/2})^2}$$

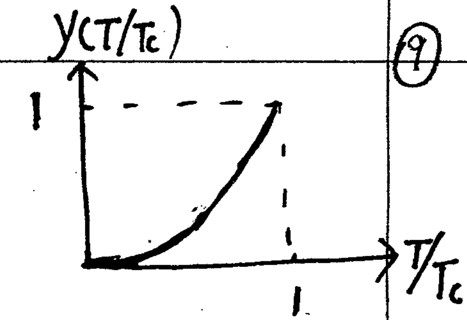
$$\frac{2}{Vol}\sum_k \to N_F \int d\xi \int \frac{d\varphi}{2\pi}$$

Define Yoshida function $Y(\varphi,T) = \frac{\beta}{4}\int_{-\infty}^{+\infty} \frac{d\xi}{\left(\cosh\frac{E(\varphi)}{2T}\right)^2}$

$$\Rightarrow \frac{\chi}{Vol} = \mu_B^2 N_F \int \frac{d\varphi}{2\pi} Y(\varphi,T)$$

For the s-wave case, $\frac{\chi}{\chi_n} = \frac{\beta}{4}\int_{-\infty}^{+\infty} \frac{d\xi}{\left[\cosh\left(\frac{(\xi^2+\Delta^2)^{1/2}}{2T}\right)\right]^2} = \frac{\beta}{2}\int_0^{+\infty} d\xi \, \mathrm{sech}^2 \frac{\beta E}{2}$

$= Y(T)$ isotropic case




at $T = T_c$, $y(1) = \int_0^{+\infty} \operatorname{sech}^2 x \, dx = 1$

$T \ll \Delta$, $y(T/\Delta)$ is suppressed exponentially.

$\sim e^{-\Delta/T}$

Now let us consider the d-wave case:

$$\frac{\chi}{\chi_n} = \int_0^{+\infty} d\xi \, \frac{\beta}{2} \frac{1}{\cosh^2\left(\frac{\xi}{2T}\right)} \int_{-\frac{\xi}{2\Delta}}^{\frac{\xi}{2\Delta}} d\,\Delta\varphi + O(e^{-\Delta/T})$$

the low T contribution from $\xi > \Delta \cos 2\varphi$

$\Rightarrow |\Delta\varphi| < \frac{\xi}{2\Delta}$

$$\simeq \int_0^{+\infty} d\xi \, \frac{1}{2k_BT} \frac{\xi}{\Delta} \frac{1}{\cosh^2\left(\frac{\xi}{2k_BT}\right)}$$

define $x = \frac{\xi}{2k_BT}$

$$\Rightarrow \boxed{\frac{\chi}{\chi_n} \sim \frac{2k_BT}{\Delta} \int_0^{+\infty} dx \, \frac{x}{\cosh^2(x)} \simeq \text{const.} \, \frac{k_BT}{\Delta}}$$

This is also consistent with the low T DOS $\sim N_F \frac{\omega}{\Delta}$.

$\chi$ can be measured through NMR knight shift. The NMR frequency of nuclear in solids is different ~~to~~ from that in vaccum: $B_{eff} = B_{ex} + B_{mol}$; and $B_{mol} \propto M = B_{ex}\chi$.

From the frequency shift (Kight shift), we can infer the magnetic susceptibility of the envirnment, i.e. electronic structure.

Normal density / superfluid dencity

We first consider Galiean invariant system. Suppose the walls of the superconductor is moving with a small velocity $\vec{v}$. Let the system be in the thermal equalibrium. The normal state density is defined as $\frac{1}{m}\vec{P}/\text{vol} = \rho_n \vec{v}$.

In other word, the normal density is a susceptibility corresponding to a moving velocity $\vec{v}$. It shifts the excitation energy $E_p \to E_p - \vec{v}\cdot\vec{p}$. And we need to translate it the language of EM response

$$\vec{J}(r) = \sum \vec{j}_i = \sum_i \frac{e}{m} (\vec{P}_i - \frac{e}{c}\vec{A}(\vec{r}_i))\, \delta^{(3)}(\vec{r}-\vec{r}_i)$$

take average $\rightarrow$ $\langle \vec{J}(r)\rangle = \frac{e}{m}\left(\langle\vec{P}\rangle - \frac{ne}{c}\vec{A}\right)$ $\quad$ $n$ is the total density

Define $\frac{e}{m}\langle\vec{P}\rangle = \frac{e}{m} \underset{\sim}{\frac{e\vec{A}}{c}} \rho_n \Rightarrow$

$$\langle \vec{J}(r)\rangle = -\frac{e}{m}(n-\rho_n)\frac{e\vec{A}}{c} \quad \text{and} \quad \boxed{\rho_s = n - \rho_n}$$

hence $\left(\frac{1}{m}\right)\frac{e}{c}\vec{A}$ plays the role of $\vec{v}$.

⊛ Calculation of $\rho_n$

for the 4 states $|\phi_k^G\rangle = u_k|0\rangle + v_k|1_\uparrow 1_\downarrow\rangle$

$$|\phi_k^{EP}\rangle = -v_k^*|0\rangle + u_k^*|1_\uparrow 1_\downarrow\rangle$$

these two states have zero mementum, They do not exihit energy shift. The two broken pair states $C_{k\uparrow}^+|0\rangle$, $C_{-k\downarrow}^+|0\rangle$

Theire energy is shifted by $\pm\hbar\vec{k}\cdot\vec{v}$

$$\Rightarrow \frac{1}{m}\vec{P} = \sum_k \frac{\hbar}{m}\vec{k}\left\{\frac{e^{-\beta(E_k-\hbar\vec{k}\cdot\vec{v})}}{(1+e^{-\beta E_k})^2} - \frac{e^{-\beta(E_k+\hbar\vec{k}\cdot\vec{v})}}{(1+e^{-\beta E_k})^2}\right\}$$

$1+2e^{-\beta E_k}+e^{-2\beta E_k}$ ( sub-Hilbert space partition)

$$= \sum_k \frac{\hbar}{m}\vec{k}\,\frac{e^{-\beta E_k}}{(1+e^{-\beta E_k})^2}\,2\beta(\hbar\vec{k}\cdot\vec{v})$$

$$= \hbar^2\sum_k\left(\frac{1}{m}\vec{k}\right)(\vec{k}\cdot\vec{v})\,\frac{\beta}{2}\,\mathrm{sech}^2\frac{\beta}{2}E_k$$

set $\vec{v} \parallel \hat{x}$, the $\frac{1}{m}\vec{P}$ is along $\hat{x}$ $\Rightarrow$

$$\frac{1}{Vol}\frac{1}{m}\vec{P} = \hbar^2\frac{1}{Vol}\sum_k \frac{k^2}{m}\,v\cdot\cos^2\varphi_k\,\frac{\beta}{2}\,\mathrm{sech}^2\frac{\beta}{2}E_k$$

$$= \hbar^2 k_f^2\left(\frac{1}{m}\right)\frac{N_F}{2}\int\frac{d\varphi}{2\pi}\cos^2\varphi_k\int_{-\infty}^{+\infty}d\xi\,\frac{\beta}{2}\,\mathrm{sech}^2\frac{\beta}{2}E_k$$

spin degrees of freedom $\quad \frac{2}{Vol}\sum_k \to N_F\int d\xi\int\frac{d\varphi}{2\pi}$

$$\Rightarrow \frac{1}{Vol}\frac{\vec{P}}{m} = \frac{\hbar^2 k_f^2}{2m}N_F\,Y(T/T_c)\,\vec{v}$$

$$\frac{\hbar^2 k_f^2}{2m} N_F = n \text{ , the total density}$$

At $\Delta = 0$, $E_k = \varepsilon_k$

overall shift of $\frac{1}{m}\Delta \vec{k} = \vec{v}$

$$\Rightarrow \quad \rho_s = n\left(1 - Y(T/T_c)\right)$$

At $T \ll \Delta$, for the d-wave case $\rho_s \simeq n\left(1 - \frac{(2\ln 2) k_B T}{\Delta}\right)$

since $\frac{1}{\lambda^2} \propto \rho_s$ i.e

$$\boxed{\begin{aligned} \lambda(T)/\lambda(0) &\sim \left(1 - \frac{2\ln 2 \cdot k_B T}{\Delta}\right)^{-1/2} \\ &= 1 + \ln 2 \cdot \frac{k_B T}{\Delta} \end{aligned}}$$

$$\boxed{\begin{aligned} &\text{For s-wave} \\ &\lambda(T)/\lambda(0) \sim \left(1 - \# e^{-\Delta/k_B T}\right)^{-1/2} \\ &\qquad \sim 1 + \# e^{-\Delta/k_B T} \end{aligned}}$$

If for lattice system $\quad \left(\frac{1}{m}\right)_{\mu\nu} \to \frac{1}{\hbar^2}\frac{\partial \varepsilon}{\partial k_\mu \partial k_\nu}$

$$J_\mu(\vec{r}) = e\sum_i \left( v_\mu \delta^{(3)}(\vec{r} - \vec{r}_i) - \frac{e}{c}\left(\frac{1}{m}\right)_{\mu\nu} A_\nu \delta^{(3)}(\vec{r} - \vec{r}_i) \right)$$

$$\langle J_\mu \rangle = e\left[ \langle v_\mu \rangle \rho_s - \frac{e}{c}\left\langle \frac{1}{m} \right\rangle_{\mu\nu} n A_\nu \right]$$

$$\left\langle \frac{1}{m} \right\rangle_{\mu\nu} = \frac{1}{V}\sum_k \left\langle \frac{\partial^2 \varepsilon}{\partial k_\mu \partial k_\nu} \right\rangle \frac{1}{\hbar^2} \longrightarrow \frac{1}{m}\delta_{\mu\nu}$$

for parabollic band

sum of spin

$$K_{\mu\nu} = -\frac{1}{c}\left[ \frac{2e^2}{V}\sum_k \frac{1}{\hbar^2}\left\langle \frac{\partial^2 \varepsilon}{\partial k_\mu \partial k_\nu} \right\rangle + \Pi_{\mu\nu}(q \to 0, \omega = 0) \right]$$

$$J_\mu = K_{\mu\nu} A_\nu \longrightarrow J_\mu(q,\omega) = K_{\mu\nu}(q,\omega) A_\nu(q,\omega)$$

Define via linear response

$$\Pi_{\mu\nu}(q, t-t') = -i\frac{\Theta(t-t')}{V}\langle|[j_\mu(+q,t),\ j_\nu(\bar{q},t')]|\rangle$$

$\xrightarrow{\text{Fourier}}$

$$\Pi_{\mu\nu}(q,\omega) = \frac{-i}{V}\int_0^{+\infty} dt\ e^{i(\omega+i0^+)t}\langle|[j_\mu(+q,t).\ j_\nu(\bar{q},0)]|\rangle$$

Mean-field

$$\Pi_{\mu\nu}(q\to 0, \omega=0) = \frac{2e^2}{V\hbar^2}\sum_k \frac{\partial\varepsilon}{\partial k_\mu}\frac{\partial\varepsilon}{\partial k_\nu}\frac{f(E_k)-f(E_{k+q})}{E_k - E_{k+q}}$$

$$= \frac{2e^2}{V\hbar^2}\sum_k \frac{\partial\varepsilon}{\partial k_\mu}\frac{\partial\varepsilon}{\partial k_\nu}\frac{\partial f}{\partial E}$$

(2) Now consider optical conductivity (finit frequecy)

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} \Rightarrow E(\omega) = \frac{i\omega}{c}A(\omega)$$

$$J_\mu = \frac{c}{i\omega}K_{\mu\nu}E_\nu \Rightarrow \sigma_{\mu\nu}(q,\omega) = \lim_{\omega\to 0}\frac{c}{i(\omega+i0^+)}K_{\mu\nu}$$

$$\sigma_{\mu\nu}(q,\omega) = \frac{i}{\omega+i0^+}\left[\frac{2e^2}{V}\sum_k\frac{1}{\hbar^2}\left\langle\frac{\partial^2\varepsilon}{\partial k_\mu\partial k_\nu}\right\rangle + \Pi_{\mu\nu}(q,\omega)\right]$$

$$\frac{1}{x+i0^+} = P\left(\frac{1}{x}\right) - i\pi\delta(x)$$

$$\mathrm{Re}\,\sigma_{\mu\nu} = \left[\frac{2\pi e^2}{V}\sum_k\frac{1}{\hbar^2}\left\langle\frac{\partial^2\varepsilon}{\partial k_\mu\partial k_\nu}\right\rangle + \pi\,\mathrm{Re}\,\Pi_{\mu\nu}\right]\delta(\omega)$$

$$-\frac{\mathrm{Im}\,\Pi_{\mu\nu}(q,\omega)}{\omega}$$

K-K relation

– spectru Representation

$$\Pi_{\mu\nu}(q,\omega) = -\frac{1}{V\hbar^2 Z}\sum_{nm}\frac{e^{-\beta E_m} - e^{-\beta E_n}}{\omega + E_n - E_m + i0^+}\langle n|J_\mu(q)|m\rangle\langle m|J_\nu(-q)|n\rangle$$

$$\int_{-\infty}^{+\infty} d\omega \, \frac{\operatorname{Im} \Pi_{\mu\nu}(q,\omega)}{\omega} = \pi \operatorname{Re} \Pi_{\mu\nu}(q,0)$$

$$\Rightarrow \int_{-\infty}^{\infty} d\omega \, \operatorname{Re} \sigma_{\mu\mu}(q,\omega) = \frac{2\pi e^2}{V\hbar^2} \sum_k \left\langle \frac{\partial^2 \xi_k}{\partial k_\mu \partial k_\nu} \right\rangle$$

At $q \to 0$, optical

$$\Rightarrow \int_0^{\infty} d\omega \, \operatorname{Re} \sigma_{\mu\mu}(\omega) = \frac{\pi e^2}{V\hbar^2} \sum_k \left\langle \frac{\partial^2 \xi_k}{\partial k_\mu \partial k_\nu} \right\rangle$$

$\xrightarrow{\text{parabolic band}}$ $\dfrac{\pi}{2} \dfrac{e^2}{m} n$

For superconductor, $j = -\dfrac{c}{4\pi\lambda_L^2} A$, $E = -\dfrac{1}{c}\dfrac{\partial A}{\partial t}$

$$\operatorname{Im} \sigma(\omega) = c^2/4\pi\lambda\omega$$

$$\operatorname{Im} \sigma(\omega) = -P \int_{-\infty}^{+\infty} \frac{d\omega'}{\pi} \frac{\operatorname{Re}\sigma(\omega')}{\omega' - \omega} \leftarrow \operatorname{Re}\sigma(\omega) = \frac{c^2}{4\lambda_L^2}\delta(\omega)$$

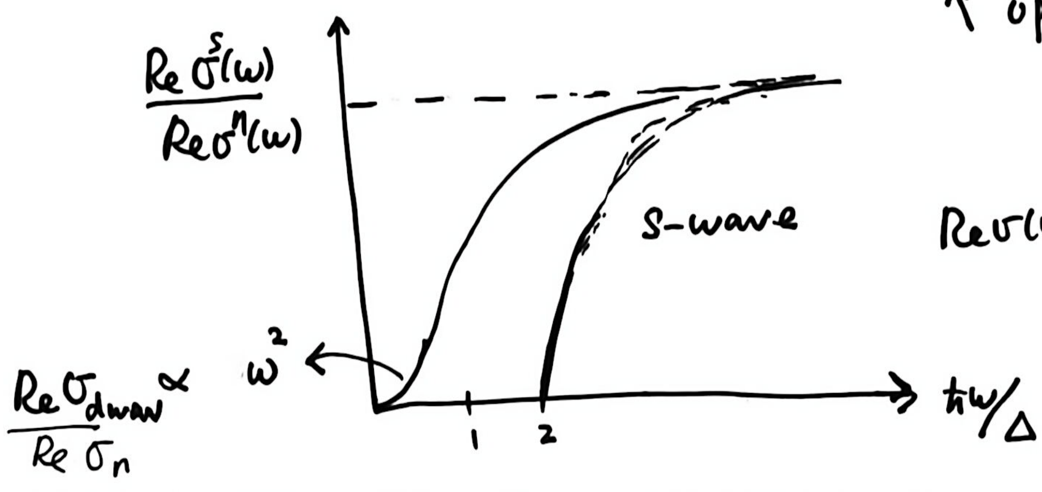
superfluid

Drude peak. $\dfrac{1}{\lambda_L^2} = \dfrac{4\pi n_s e^2}{mc^2}$

$$\Rightarrow \boxed{\int_{0^+}^{\infty} d\omega \, \operatorname{Re} \sigma_\mu(\omega) = \frac{\pi}{2}\frac{e^2}{m}(n - n_s^\mu)}$$

superfluid contribution

optical sum rule




$$\operatorname{Re}\sigma(\omega) \propto \frac{1}{\omega}\int_0^\omega dE \, \rho(E)\rho(\omega - E)$$

$$= \frac{1}{\omega}\int_0^\omega dE \, E(\omega - E)$$

$$\propto \omega^2$$

Phase - sensitive measurment — d-wave symmetry

The therodynamic anomalies of the d-wave superconductors only detect the linear density of states of nodal quasi-particles. They are not phase sensitive — we need smoking gun evidence for sign-change of gap functions. Below we will see this from Josephson tunneling junction.

According to linear-response theory, the tunneling currents between SCs (see Dan's notes)

$$A = \sum_{kq\sigma} T_{kq} C^{\dagger}_{L,k\sigma} C_{Rq\sigma}, \rightarrow A(t) = \sum_{kq\sigma} T_{kq} C^{\dagger}_{L,k\sigma}(t) C_{Rq\sigma}(t).$$

$$K_0 = H_0 - \mu_L N_L - \mu_R N_R, \text{ and } eU = \mu_L - \mu_R$$

$$\rightarrow C_{\alpha,k\sigma}(t) = e^{-iK_0 t} C_{\alpha,k\sigma} e^{iK_0 t}, \quad \alpha = R, L.$$

The tunneling currents $I(t) = I_Q(t) + I_J(t)$

$$I_Q(t) = -\frac{2e}{\hbar^2} \mathrm{Im} \int_{-\infty}^{+\infty} dt' \, e^{ieU(t-t')} X_{ret}(t-t') \leftarrow \text{normal current}$$

$$I_J(t) = \frac{2e}{\hbar^2} \mathrm{Im} \int_{-\infty}^{+\infty} dt' \, e^{-ieU(t+t')} Y_{ret}(t-t') \leftarrow \text{Josephson tunneling}$$

retarded Green's function

$$X_{ret}(t-t') = -i\theta(t-t') \langle [A(t), A^{\dagger}(t')] \rangle_0$$

$$Y_{ret}(t-t') = -i\theta(t-t') \langle [A^{\dagger}(t) \, A^{\dagger}(t')] \rangle_0 \leftarrow \text{tunneling } L \rightarrow R.$$

⇒ The Josephson channel

$$I_J(t) = \frac{2e}{\hbar^2} \mathrm{Im} [ e^{-2ieUt} Y_{ret}(\Omega = eU) ]$$

where $y_{ret}(\Omega) = \int_{-\infty}^{+\infty} dt \; e^{i\Omega t} \; y_{ret}(t)$

$$y_{ret}(\Omega) = e^{i(\phi_R + \phi_L)} \sum_{kq} T_{kq} T_{-k-q} \frac{\Delta_L(k)}{E_L(k)} \cdot \frac{\overset{*}{\Delta_R}(q)}{E_R(q)}$$

$$\left\{ \frac{1}{\hbar\Omega + E_L(k) + E_R(q) + i\eta} - \frac{1}{\hbar\Omega - E_L(k) - E_R(q) + i\eta} \right] .$$

* Check dimesion $[I] = \frac{e}{\hbar^2} \cdot [time] \cdot [energy]^2 = \frac{e}{\hbar} \, e \, [Volt] = \frac{e^2}{\hbar} \cdot [Voltage]$

correct : $\frac{e^2}{\hbar}$ is the unit of conductance.

Aussuming $T_{kq}$'s are momentum indepedent $\Rightarrow$

$$y_{ret}(\Omega) = \frac{\hbar^2 G_N}{2\pi e^2} e^{i(\phi_R - \phi_L)} \int_0^{+\infty} d\xi_L \int_0^{+\infty} d\xi_R \int \frac{d\varphi_L}{2\pi} \int \frac{d\varphi_R}{2\pi}$$

$$\left\{ \frac{\Delta_L(\xi_L, \varphi_L)}{E_L} \frac{\overset{*}{\Delta_R}(\xi_R, \varphi_R)}{E_R} \right\} \left[ \frac{1}{\hbar\Omega + E_L + E_R + i\eta} - \frac{1}{\hbar\Omega - E_L - E_R + i\eta} \right]$$

The new property of unconventional SC is the appeareace of angular dependence of

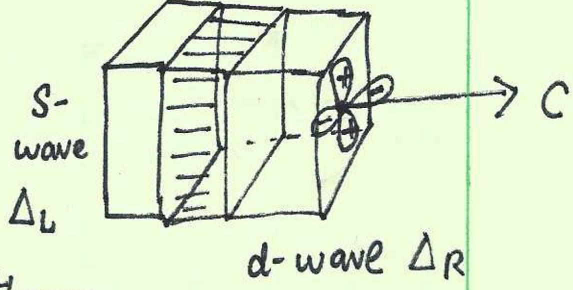
$$\boxed{\int \frac{d\varphi_L}{2\pi} \int \frac{d\varphi_R}{2\pi} \quad \Delta_L(\xi_L, \varphi_R) \overset{*}{\Delta_R}(\xi_R, \varphi_R)}$$

(if in 3d system $\int \frac{d\varphi_{L,R}}{2\pi} \rightarrow \int \frac{d\Omega_{L,R}}{4\pi}$ )

we can neglect angular dependence in $E_L$ and $E_R$, because their dependence is $|\Delta_L|^2$ and $|\Delta_R|^2$.

1 Tunneling between s-wave and d-wave SC along the c-axis

$$\int \frac{d\varphi_R}{2\pi} \Delta_R^*(\xi_R, \varphi_R) = 0 !$$




No Josephson tunneling. This result is exact up to second order perturbation theory.

Q: Effective Ginzburg-Landau equation: Can we writedown a coupling at the quadratic order?

$\boxed{\text{YBCO is a different story: it's S+d}}$

$$\Delta F = -J(\Delta_s^* \Delta_d + c.c.)$$

No! this term is not invariant under rotation 90° around c-axis.

but s and d-wave part do can couple at quartic order as

$$\Delta F = -J(\Delta_d^* \Delta_s)^2 + c.c.$$

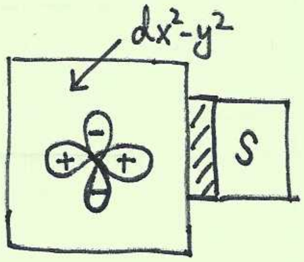
$$\rightarrow \quad I \propto \sin[2(\phi_R - \phi_L + eUt)]$$

high order Josephson effect, two-pair tunneling.

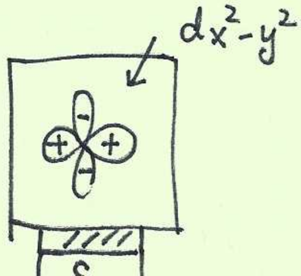
2 however, if the junction is set in the ab-plane

due to geometry, we cannot neglect the momentum dependence of $T_{k,q}$.

For example, the tunneling matrix elements for $k \parallel$ surface and $k \perp$ surface are different.



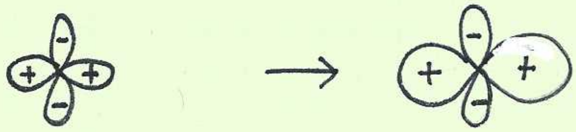

A)




B)

For example, in A) we expect that the tunneling for $\vec{k} \parallel \hat{x}$ is the much stronger than $\vec{k} \parallel \hat{y}$. Thus the d-wave gap function is not averaged evenly, i.e. $\int \frac{d\varphi_L}{2\pi} \Delta_L(\xi_L, \varphi_L)\, t(\varphi_L) \neq 0$.

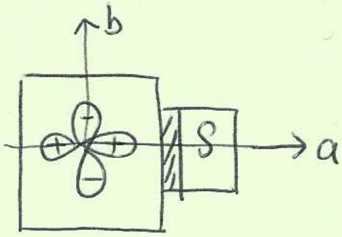
effect from tunneling matrix element.



$\Delta_L$ $\quad\quad$ $\Delta_L t(\varphi_L)$

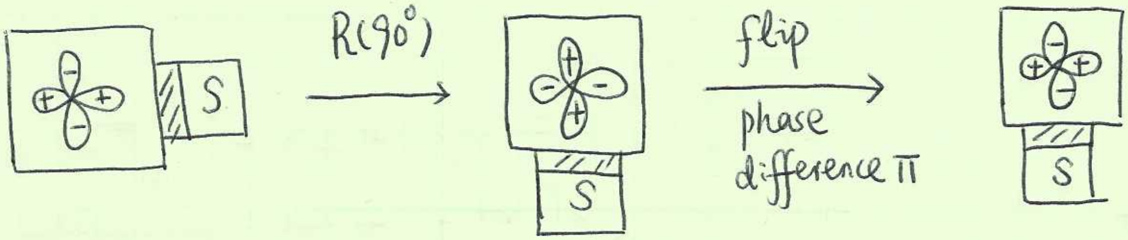
We do have coupling at quadratic level as

$$\Delta F_x = -J\, (\Delta_{s_1}^* \Delta_d + c.c).$$



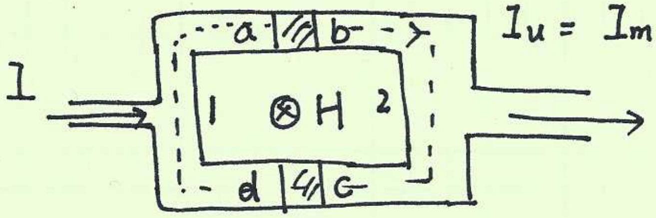

(There's no symmetry about this set-up which can change the sign of the d-wave part!)

- Nevertheless, we can build up the relation between the coupling configurations of A) and B).




$$\Delta F_y = -J\, (\Delta_{s_2}^* \Delta_d + c.c)$$

Digression ①: coupled two-junction SQUID




$I_u = I_{mu} \sin \tilde{\Delta}\phi_u$

$I_d = I_{md} \sin \tilde{\Delta}\phi_d$

$$I = I_u + I_d = I_{mu} \sin \tilde{\Delta}\phi_u + I_{ml} \sin \tilde{\Delta}\phi_d$$

should be gauge invariant phase difference

If $I_{mu} = I_{ml} = I_m$, we say that these two junctions are matched.

$$\Rightarrow \quad I = 2 I_m \sin \frac{\tilde{\Delta}\phi_u + \tilde{\Delta}\phi_d}{2} \cos \frac{\tilde{\Delta}\phi_u - \tilde{\Delta}\phi_d}{2}$$

$$\oint \nabla\phi \cdot dl = (\phi_b - \phi_a) + (\phi_c - \phi_b) + (\phi_d - \phi_c) + (\phi_a - \phi_d) = 2n\pi$$

The phase difference across the up and down junction

gauge invariant phase: $\boxed{\tilde{\Delta}\phi_u = \phi_b - \phi_a - \left(\int_a^b \vec{A} \cdot dl\right) 2\pi / \Phi_0}$

that enters the formula of current.

$$\Rightarrow \quad \phi_b - \phi_a = \tilde{\Delta}\phi_u + \left(\int_a^b \vec{A} \cdot d\vec{l}\right) \cdot \frac{2\pi}{\Phi_0}$$

$$\phi_d - \phi_c = -\tilde{\Delta}\phi_d + \left(\int_c^d \vec{A} \cdot d\vec{l}\right) \frac{2\pi}{\Phi_0}$$

(phase arcoss the junction).

$\Phi_0 = \frac{hc}{2e} = 2.07 \times 10^{-7}$ Gauss. $cm^2$

$$\phi_c - \phi_b = \int_b^c \nabla\phi \cdot dl = \frac{2\pi}{\Phi_0} \int_b^c \left(\vec{A} + \frac{4\pi}{c} \lambda_L^2 \vec{j}\right) \cdot d\vec{l}$$

inside superconductor

$$\phi_a - \phi_d = \int_d^a \nabla\phi \cdot dl = \frac{2\pi}{\Phi_0} \int_d^a \left(\vec{A} + \frac{4\pi}{c} \lambda_L^2 \vec{j}\right) \cdot d\vec{l}$$

Add ~~together~~ $\Rightarrow$ $\Delta\phi_u - \Delta\phi_d + \oint \vec{A}\cdot d\ell \left(\frac{2\pi}{\Phi_0}\right) + \frac{2\pi}{\Phi_0}\frac{4\pi\lambda_L^2}{c}\int_{c'} \vec{j}\cdot d\vec{\ell}$

$= 2n\pi$

$\Rightarrow \Delta\phi_u - \Delta\phi_d = -\frac{2\pi}{\Phi_0}\oint \vec{A}\cdot d\vec{\ell} - \frac{2\pi}{\Phi_0}\frac{4\pi\lambda_L^2}{c}\int_{c'} \vec{j}\cdot d\vec{\ell}$ ↖ exclude the insulater junction

we can choose the loop deep inside the superconductor, such that $j = 0$ $\Rightarrow$

$$\Delta\phi_u - \Delta\phi_d = -\frac{2\pi}{\Phi_0}\oint \vec{A}\cdot d\vec{\ell} = -2\pi\,\Phi/\Phi_0$$

$$\Rightarrow I = 2I_m \sin\left(\Delta\phi_u + \pi\,\Phi/\Phi_0\right)\cos\left(\frac{\pi\Phi}{\Phi_0}\right) \quad ①$$

If the inductance of the loop is considered, the flux $\Phi$ consists two part: $\Phi = \Phi_{ex} + L I_{cir}$ ②

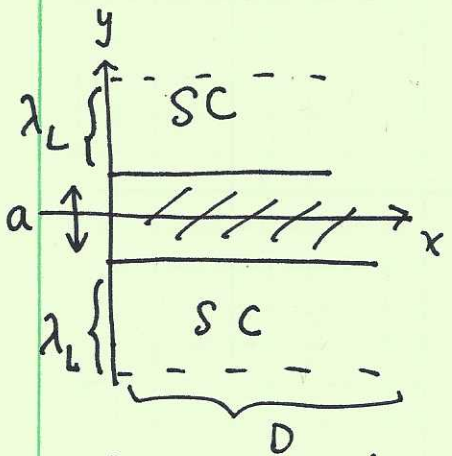
and the circulate current $I_{cir} = I_m(\sin\Delta\phi_u - \sin\Delta\phi_d)$ ③

In principle ①, ②, ③ should be solved consistently. If the self-inductance can be neglected, we have

$$I = 2I_m \sin\left(\Delta\phi_u + \pi\frac{\Phi_{ex}}{\Phi_0}\right)\cos\left(\frac{\pi\Phi_{ex}}{\Phi_0}\right),$$

$\Rightarrow I_{max} = 2I_m \left|\cos\frac{\pi\Phi_{ex}}{\Phi_0}\right|$, the maximum supercurrent desity oscillatate with $\Phi_{ex}/\Phi_0$.

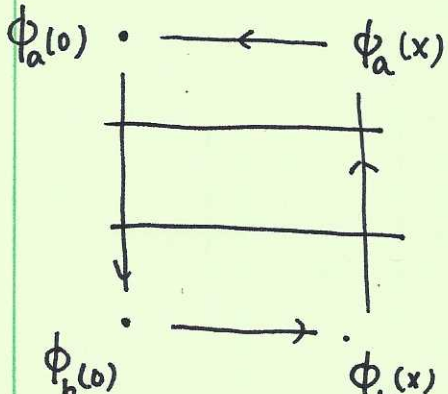
Digression ②: Fraunhofer pattern : a single planar Josephson junction.




$$A_x = 0,$$

$$A_y = \begin{cases} -Bx\, e^{-(y-a/2)/\lambda_L} & y > a/2 \\ -Bx & a/2 > y > -a/2 \\ -Bx\, e^{(y+a/2)/\lambda_L} & y < -a/2 \end{cases}$$

the gauge-independent phase difference




$$\Rightarrow \quad \tilde{\Delta}\phi(x) = \tilde{\Delta}\phi(0) - \frac{2e}{\hbar c}\int_{-\infty}^{+\infty} A_y(x)\, dy$$

$$= \tilde{\Delta}\phi(0) - \frac{2e}{\hbar c} B(a+2\lambda_L)\, x$$

$$\left\{ \begin{aligned} \oint \nabla\phi\, dl = &\left(\phi_a(x) - \phi_b(x)\right) + \left(\phi_a(0) - \phi_a(x)\right) \\ &+ \left(\phi_b(0) - \phi_a(0)\right) + \left(\phi_b(x) - \phi_b(0)\right) = 2n\pi \end{aligned} \right.$$

$$\left.\begin{aligned} \tilde{\Delta}\phi_a(x) &= \phi_a(x) - \phi_b(x) - \frac{2\pi}{\Phi_0}\int_{bx}^{ax} \vec{A}\cdot d\vec{l} \\ \tilde{\Delta}\phi(0) &= \phi_a(0) - \phi_b(0) - \frac{2\pi}{\Phi_0}\int_{b0}^{a0} \vec{A}\cdot d\vec{l} \\ \Rightarrow \quad \phi_a(0) - \phi_a(x) &= -\frac{2\pi}{\Phi_0}\int_{ax}^{a0} \vec{A}\cdot d\vec{l} \quad \cdots \\ \phi_b(x) - \phi_b(0) &= -\frac{2\pi}{\Phi_0}\int_{b0}^{bx} \vec{A}\cdot d\vec{l} \end{aligned}\right\}$$

⑧

$$\Rightarrow I = \int_0^D j_y(x)\,dx = j_m \int_0^D dx\, \sin(\tilde{\Delta}\phi(x))$$

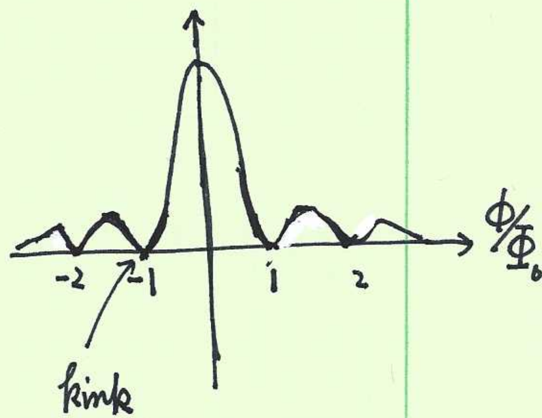
$$= j_m D \int_0^1 d\,x/_D\ \sin\left(\tilde{\Delta}\phi(0) - \frac{2e}{\hbar c} BD(a+2\lambda_L)\, x/_D\right) \quad \text{define } \Phi = BD(a+2\lambda_L)$$

$$= j_m D \int_0^1 dx'\ \sin\left(\tilde{\Delta}\phi(0) - 2\pi\frac{\Phi}{\Phi_0}\,x'\right)$$

$$= j_m D\, \frac{1}{2\pi\,\Phi/\Phi_0}\left[\cos\left(\tilde{\Delta}\phi(0) - \frac{2\pi}{\Phi_0}\Phi\right) - \cos(\tilde{\Delta}\phi(0))\right]$$

$$= j_m D\, \frac{\sin \pi\,\Phi/\Phi_0}{\pi\Phi/\Phi_0} \cdot \sin\left(\tilde{\Delta}\phi(0) - \frac{\pi\Phi}{\Phi_0}\right)$$

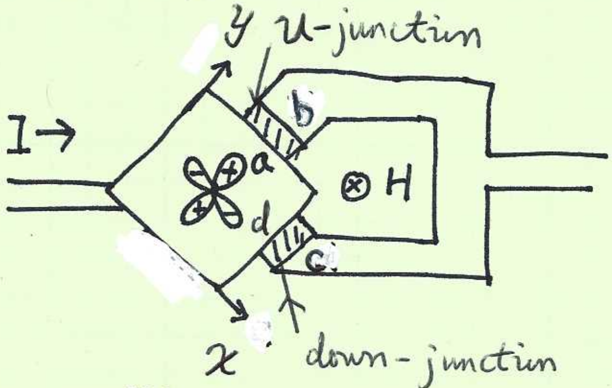
$$\Rightarrow I_{max} = j_m D \left|\frac{\sin \pi\,\Phi/\Phi_0}{\pi\,\Phi/\Phi_0}\right|$$



Now let us apply it to the d-wave case

① d-S SQUID:

The two junctions have opposit sign of coupling constants




$$I = I_u + I_d = I_m\left(\sin\tilde{\Delta}\phi_u - \sin\tilde{\Delta}\phi_d\right)$$

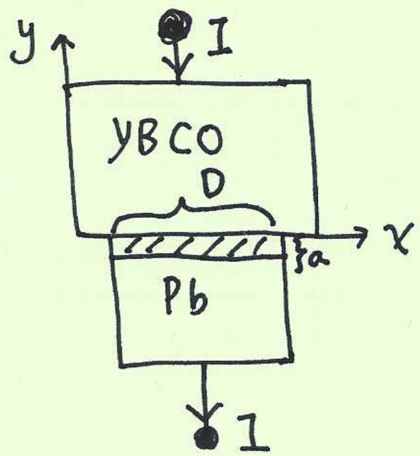
$$= 2I_m \sin\frac{\tilde{\Delta}\phi_u - \tilde{\Delta}\phi_d}{2}\cos\frac{\tilde{\Delta}\phi_u + \tilde{\Delta}\phi_d}{2} = 2I_m \sin\frac{\pi\Phi}{\Phi_0}\cos\left(\Delta\phi_u + \frac{\pi\Phi}{\Phi_0}\right)$$

$$\Rightarrow \boxed{I_{max} = 2I_m\left|\sin\pi\Phi/\Phi_0\right|}$$

The d-S SQUID has maximum current density at $\Phi = \frac{1}{2}\Phi_0$ due to geometric flux $\pi$.

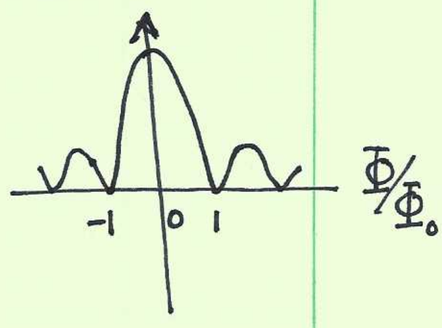
② Conner junction

if we make a planar Josephsn junction




This planar junction will exhibit the same Fraunhofer pattern as two s-wave one, i.e

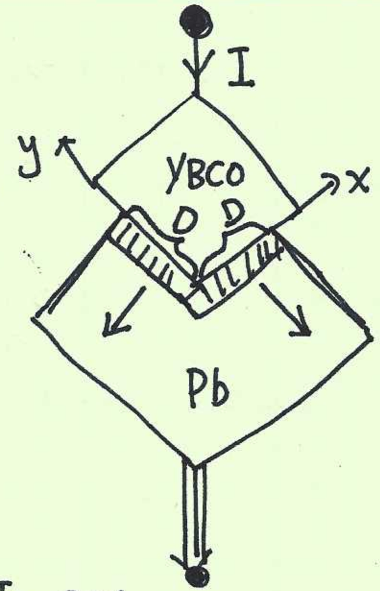
$$I_{max} \propto \left|\frac{\sin\pi\Phi/\Phi_0}{\pi\Phi/\Phi_0}\right|$$



define $L = a + \lambda_{YBCO} + \lambda_{Pb}$

a: is the width of insulating regin

L: is effective width of junctin




for x-direction junctin, its current projectin to 45° -directin is

$$\frac{1}{\sqrt{2}} j_m \int_0^1 dx' \sin\left(\tilde{\Delta}\phi(0) - \frac{2\pi\Phi}{\Phi_0} x'\right) \qquad \Phi = BDL$$

when calculate the current to y-directin, we need to change gauge

$$A_y = 0 \ , \ A_x = By \qquad \left(\text{c.f. the odd gauge } \begin{matrix} A_x = 0 \\ A_y = -Bx \end{matrix}\right).$$

$\Rightarrow$ current along 45° : $-\frac{1}{\sqrt{2}} j_m \int_0^1 dy' \sin\left(\tilde{\Delta}\phi(0) + \frac{2\pi\Phi}{\Phi_0} y'\right)$

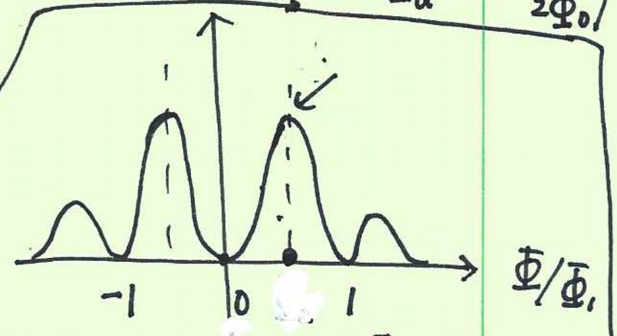
the overall "−" sign comes from the d-wave symmetry

$$\Rightarrow I_{tot} = \frac{j_m}{\sqrt{2}} D \frac{\sin \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0} \left[\sin\left(\tilde{\Delta}\phi(0) - \frac{\pi\Phi}{\Phi_0}\right) - \sin\left(\tilde{\Delta}\phi(0) + \frac{\pi\Phi}{\Phi_0}\right)\right]$$

$$= -\sqrt{2} j_m D \frac{\sin^2 \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0} \cos(\tilde{\Delta}\phi(0))$$

$$\frac{\pi\Phi_a}{\Phi_0} = \tan\left(\frac{\pi\Phi_a}{2\Phi_0}\right)$$

$$\Rightarrow \boxed{I_{max} = I_0 \frac{\sin^2 \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0}}$$




$\Phi_a = 0.742\Phi_0$