Lecture 4 — Extensions

1. Multi-fermion clustering instabilities

$\Delta_1^\dagger = \psi_{1\uparrow}^\dagger \psi_{1\downarrow}^\dagger \propto e^{i\sqrt{\pi}\theta_1}$ $\qquad$ $\Delta_2^\dagger = \psi_{2\uparrow}^\dagger \psi_{2\downarrow}^\dagger \propto e^{i\sqrt{\pi}\theta_2}$

$\Delta_1^\dagger \Delta_2^\dagger \propto e^{i\sqrt{4\pi}\theta_+}$ $\qquad$ $\Delta_1^\dagger \Delta_2 \propto e^{i\sqrt{\pi}\theta_-}$

(Quartetting / 4e) $\qquad$ (high order normal state)

2. Frustrated superfluidity — sextetting order

3-coloring model.

3 Intra-orbital pairing →

pair density wave

4 Hund's rule assisted pairing

$La_3 Ni_2 O_7$

1. Multi-particle states

- Nuclear physics: α- particle, $\begin{matrix} p & n \\ p & n \end{matrix}$ spin & isospin singlet

- QCD: baryon, SU(3) color singlet

  tetra-quark $uc\bar{s}\bar{d}$, $\bar{u}c\bar{s}d$, penta-quark $uc\bar{c}sd$

- Condensed matter: biexciton

★ Difficulty: lack of a BCS-type well controlled mean-field theory

★ Field theoretical menthod 1+1D. C. Wu, PRL 95, 266404 (2005)

- variational WF, Jiang & Hu, arxiv: 2209.13905

  X. Cui, PRL 131, 193401 (2023)

- particle-hole channel: 2-particle / 2-hole

$$\langle c^\dagger c^\dagger d d \rangle \quad , \quad v.s \quad \langle c^\dagger c^\dagger d^\dagger d^\dagger \rangle$$

2. 2-band SC

$$\Delta_1^\dagger = \psi_{1\uparrow}^\dagger \psi_{1\downarrow}^\dagger \propto e^{i\sqrt{\pi}\theta_1}, \quad \Delta_2^\dagger = \psi_{2\uparrow}^\dagger \psi_{2\downarrow}^\dagger \propto e^{i\sqrt{\pi}\theta_2}$$

Define overall and relative phases

$$\theta_+ = (\theta_1+\theta_2)/2, \quad \theta_- = \theta_1 - \theta_2$$

Then
$$\Delta_1^\dagger \Delta_2^\dagger = \psi_{1\uparrow}^\dagger \psi_{1\downarrow}^\dagger \psi_{2\uparrow}^\dagger \psi_{2\downarrow}^\dagger \propto e^{i\sqrt{4\pi}\theta_+}$$
$$\Delta_1^\dagger \Delta_2 = \psi_{1\uparrow}^\dagger \psi_{1\downarrow}^\dagger \psi_{2\uparrow} \psi_{2\downarrow} \propto e^{i\sqrt{\pi}\theta_-}$$

θ_+ \ θ_-	ordered	disordered
ordered	2-band SC	quartetting
disorder	high-order $\downarrow$ normal	normal

3 A concrete model, C. Wu PRL 95, 266404 (2005)
C. Wu et al PRL 91, 186402 (2003)

4-component fermion $S = 3/2$, $S_z = \pm 1/2, \pm 3/2$

$$\Delta_1^\dagger = \psi_{3/2}^\dagger \psi_{-3/2}^\dagger \qquad \Delta_2^\dagger = \psi_{1/2}^\dagger \psi_{-1/2}^\dagger$$

Singlet (: intra-band) $\eta^\dagger(r) = \Delta_1^\dagger - \Delta_2^\dagger$

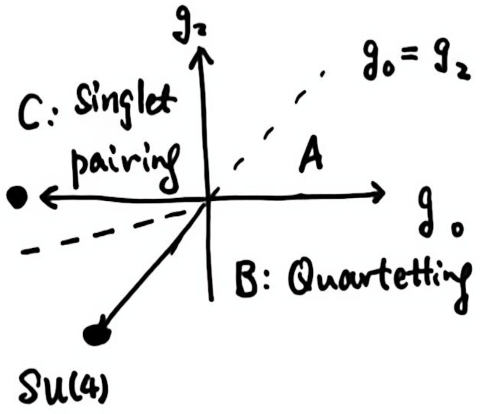
Quintet (inter-band) $\chi_0 = \psi_{3/2}^\dagger \psi_{1/2}^\dagger \pm \psi_{-1/2}^\dagger \psi_{-3/2}^\dagger$

$$\psi_{3/2}^\dagger \psi_{-1/2}^\dagger \pm \psi_{1/2}^\dagger \psi_{-3/2}^\dagger$$

$$\psi_{3/2}^\dagger \psi_{-3/2}^\dagger + \psi_{1/2}^\dagger \psi_{-1/2}^\dagger$$

$$H_{int} = \frac{g_0}{2} \eta^\dagger(r)\eta(r) + \frac{g_2}{2} \sum_{a=1\sim5} \chi_a^\dagger(r)\chi_a(r)$$

phase digram



phase A: Luttinger liquid

- superfluid phases
  B quartetting phase
  C pairing phase
- Quaintetting pairing correlation decays exponentially

$$\eta^\dagger = \Delta_1^\dagger - \Delta_2^\dagger \propto e^{i\sqrt{\pi}\theta_+} \cos\sqrt{\pi}\theta_-$$

$$Q^\dagger = \Delta_1^\dagger \Delta_2^\dagger \propto e^{i\sqrt{4\pi}\theta_+} \cos\sqrt{\pi}\phi_-$$

$\sqrt{\pi}\phi_-$: dual field of the relative phase $\sqrt{\pi}\theta_-$.

charge conservation → θ_+ power-law

$$\langle e^{i\sqrt{\pi}\theta_+(x)} e^{-i\sqrt{\pi}\theta_+(x')} \rangle \sim \frac{1}{(x-x')^{1/2k_c}}$$

$$\langle e^{i\sqrt{4\pi}\theta_+(x)} e^{-i\sqrt{4\pi}\theta_+(x')} \rangle \sim \frac{1}{(x-x')^{2/k_c}}$$

Competition between paing phase and quarletting phase

pairing of pairs: $\Delta_1 \to -\Delta_1$, $\Delta_2 \to -\Delta_2$; $\eta \to -\eta$.

$$Q_1 = \Delta_1 \Delta_2 \to Q$$

Hence; pairing phase is a Z_2-breaking phase, and Quartetting phase is a Z_2-disordered phase.

Competition between pairing and quartetting

$$H_{eff} = \frac{1}{2\pi a}\left(\lambda_1 \cos\sqrt{4\pi}\,\theta_- + \lambda_2 \cos\sqrt{4\pi}\,\phi_-\right)$$

bosonization $\psi_R = \frac{1}{\sqrt{2\pi a}} e^{i\sqrt{4\pi}\phi_R}$ $\quad \psi_L = \frac{1}{\sqrt{2\pi a}} e^{-i\sqrt{4\pi}\phi_L}$

$$[\phi_R(x), \phi_R(x')] = i/4\ \mathrm{sgn}(x-x') \qquad [\phi_R(x), \phi_L(x')] = i/4$$

$$[\phi_L(x), \phi_L(x')] = -i/4\ \mathrm{sgn}(x-x')$$

$$[\phi_R(x), \phi_L(x')] = i/4$$

$$\psi_R^\dagger \psi_L = \frac{1}{2\pi a} e^{-i\sqrt{4\pi}\phi_R} e^{-i\sqrt{4\pi}\phi_L} = \frac{1}{2\pi a} e^{-i\sqrt{4\pi}\phi} e^{-\frac{4\pi}{2}\overset{i/4}{[\phi_R,\phi_L]}}$$

$$= -i/2\pi a\ e^{-i\sqrt{4\pi}\phi} \qquad e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$$

$$\psi_L^\dagger \psi_R = \frac{i}{2\pi a} e^{i\sqrt{4\pi}\phi}$$

$$\boxed{i(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R) = \frac{1}{\pi a}\cos\sqrt{4\pi}\phi} \leftarrow \text{CDW } 2k_f$$

$$\Delta^\dagger = :\psi_R^\dagger \psi_L^\dagger: = \frac{1}{2\pi a} e^{-i\sqrt{4\pi}\phi_R} e^{i\sqrt{4\pi}\phi_L} = \frac{1}{2\pi a} e^{-i\sqrt{4\pi}\theta} e^{4\pi[\phi_R,\phi_L]/2}$$

$$= \frac{i}{2\pi a} e^{-i\sqrt{4\pi}\theta}$$

$$\Delta = \frac{-i}{2\pi a} e^{i\sqrt{4\pi}\theta} \Rightarrow \boxed{-i(\psi_R^\dagger \psi_L^\dagger - \psi_L \psi_R) = \frac{1}{\pi a}\cos\sqrt{4\pi}\theta}$$

Cooper pairing

Define $\psi_R = \frac{\xi_R + i\eta_R}{\sqrt{2}}$, $\quad \psi_L = \frac{\xi_L + i\eta_L}{\sqrt{2}}$

$$\psi_R^\dagger \psi_L = \frac{1}{2}(\xi_R - i\eta_R)(\xi_L + i\eta_L) = \frac{1}{2}(\xi_R\xi_L + \eta_R\eta_L + i(\xi_R\eta_L - \eta_R\xi_L))$$

$$i(\psi_R^\dagger\psi_L - \psi_L^\dagger\psi_R) = i(\xi_R\xi_L + \eta_R\eta_L)$$

$$\psi_R^\dagger\psi_L^\dagger = \frac{1}{2}(\xi_R - i\eta_R)(\xi_L - i\eta_L) = \frac{1}{2}(\xi_R\xi_L - \eta_R\eta_L - i(\xi_R\eta_L + \eta_R\xi_L))$$

$$-i(\psi_R^\dagger\psi_L^\dagger - \psi_L\psi_R) = -i(\xi_R\xi_L - \eta_R\eta_L)$$

$$\Rightarrow \quad H = H_0 + H_{int,eff} = H_\xi + H_\eta$$

$$= \int dx \; -\frac{i}{2}(\xi_R\partial_x\xi_R - \xi_L\partial_x\xi_L) + i(\lambda_1 - \lambda_2)(\xi_R\xi_L)$$

$$-\frac{i}{2}(\eta_R\partial_x\eta_R - \eta_L\partial_x\eta_L) + i(\lambda_1 + \lambda_2)(\eta_R\eta_L)$$

Map to

Transverse field Ising model

$$H = \frac{-1}{2} g \sum_i \sigma_x(i) + \sigma_z(i)\,\sigma_z(i+1)$$

Jordan-Wigner
$$\xi_1(n) = \frac{1}{\sqrt{2}}\left(\prod_{i<n}\sigma_x(i)\right)\sigma_y(n)$$

$$\xi_2(n) = \frac{1}{\sqrt{2}}\prod_{i<n}\sigma_x(i)\;\sigma_z(n)$$

$$\{\xi_i(n), \xi_j(n')\} = \delta_{ij}\delta_{nn'}$$

$$\Rightarrow \quad \sigma_x(i) = -i\,\sigma_y(i)\,\sigma_z(i) = -2i\,\xi_1(i)\,\xi_2(i)$$

$$\xi_1(i)\,\xi_2(i+1) = \frac{1}{2}\left(\prod_{i'<i}\sigma_x(i')\right)\sigma_y(i)\left(\prod_{i'<n+1}\sigma_x(i')\right)\sigma_z(i+1)$$

$$= \frac{1}{2}\left(\prod_{i<i'} \sigma_x(i)\right)^2 \sigma_y(n)\, \sigma_x(n)\, \sigma_z(n+1) = -\frac{i}{2}\, \sigma_z(n)\, \sigma_z(n+1)$$

$$\Rightarrow \quad H = \sum_i ig\, \xi_1(i)\xi_2(i) - i\xi_1(i)\xi_2(i+1)$$

$$= \frac{1}{2}(1-g)\sum_i (-i\,\xi_1(i)\xi_2(i) + i\,\xi_2(i)\xi_1(i))$$

$$+ \frac{1}{2}\sum_i (-i)\,\xi_1(i)(\xi_2(i+1)-\xi_2(i)) - i\,\xi_2(i)(\xi_1(i)-\xi_1(i+1))$$

$$\to H_{eff} = \frac{1}{2}\int dx\, \frac{1-g}{a}\, (\xi_1(x), \xi_2(x)) \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$+ \frac{1}{2}\int dx\, (\xi_1\ \xi_2) \begin{pmatrix} 0 & -i\partial_x \\ i\partial_x & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

define $\xi_{R,L} = \frac{\xi_1 \pm \xi_2}{\sqrt{2}}$

$$m = \frac{1}{2a}(1-g)$$

$$H_{eff} = \int dx\, -\frac{i}{2}(\xi_1 \partial_x \xi_2 + \xi_2 \partial_x \xi_1) - im\,\xi_1\xi_2$$

$$= \int dx\, \frac{-i}{2}(\xi_R \partial_x \xi_R - \xi_L \partial_x \xi_L) + im\,\xi_R \xi_L$$

Hence,

$$H_{eff} = \frac{v}{2}\int dx\, (\partial_x\phi)^2 + (\partial_x\theta)^2$$

$$+ \frac{1}{2\pi a}(\lambda_1 \cos\sqrt{4\pi}\,\phi + \lambda_2 \cos\sqrt{4\pi}\,\theta)$$

$$\to H = v\int (\psi_R^\dagger (i\partial_x)\psi_R - \psi_L^\dagger (i\partial_x)\psi_L)\, dx$$

$$+ i\lambda_1(\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R) - i\lambda_2(\psi_R^\dagger \psi_L^\dagger - \psi_L \psi_R)$$

→ Two Ising models with massess $\lambda_1 \pm \lambda_2$.
(Majorana fermions)

- $\lambda_1 > \lambda_2$ θ_- pinned. pairing phase

$\lambda_1 < \lambda_2$ ϕ_- pinned. θ_- disordered, Quartetting phase.

⊛ Competing orders

Q v.s. $2k_f$ CDW

$$Q = \psi^\dagger_{3/2}\psi^\dagger_{1/2}\psi^\dagger_{-1/2}\psi^\dagger_{-3/2} \propto e^{i\sqrt{4\pi}\,\theta_+}$$

$$N_{2k_f} = \psi^\dagger_{R\alpha}\psi_{L\alpha} \propto e^{i\sqrt{\pi}\,\phi_+}$$

scaling dimensions

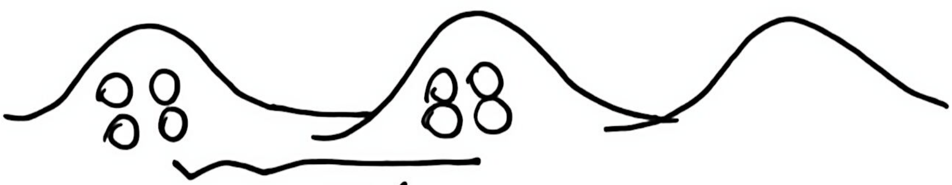
$$\Delta_Q = \frac{1}{K_c}$$

$$\Delta_{2k_f} = \frac{K_c}{4}$$

$$\langle Q(x)\, Q(x')\rangle \sim \frac{1}{(x-x')^{2\Delta_Q}} = \frac{1}{(x-x')^{2/K_c}}$$

$$\langle N_{2k_f}(x)\, N_{2k_f}(x')\rangle \sim \frac{1}{(x-x')^{K_c/2}}$$

Hence, $2k_f$ CDW wins at $K_c < 2$, and quartet wins at $K_c > 2$.



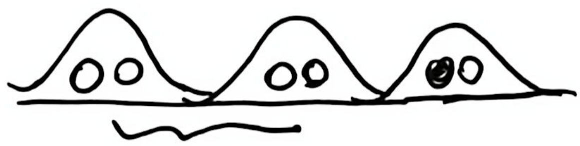
$d = 2\pi/2k_f$

pairing phase v.s. $4k_f$ CDW

$$\eta^+ = \psi^\dagger_{3/2}\psi^\dagger_{-3/2} - \psi^\dagger_{1/2}\psi^\dagger_{-1/2} \propto e^{-i\sqrt{\pi}\,\theta_+} \quad \text{wins at } K_c > 1/2$$

$$O_{4k_f\text{-CDW}} = \psi^\dagger_{R\alpha}\psi^\dagger_{R\beta}\psi_{L\beta}\psi_{L\alpha} \propto e^{i\sqrt{4\pi}\,\phi_+} \quad \text{wins at } K_c < 1/2$$

$$\Delta_\eta = \frac{\pi}{4\pi} K_c = \frac{K_c}{4}, \qquad \Delta_{4k_f\,\mathrm{CDW}} = \frac{4\pi}{4\pi}\frac{1}{K_c} = \frac{1}{K_c}$$



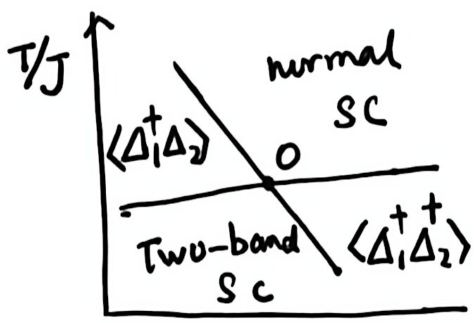
$d = 2\pi/4k_f$

⊛ 2D - classic theory

$$\Delta F = \lambda |\Delta_1|^2 |\Delta_2|^2 + \lambda' (\Delta_1^*\Delta_1^*\Delta_2\Delta_2 + \text{c.c.})$$

$$\Delta F = 2\lambda' |\Delta_1|^2 |\Delta_2|^2 \cos 2(\theta_1 - \theta_2) \rightarrow \lambda' > 0 \Rightarrow \Delta\theta = \theta_1 - \theta_2 = \pm\pi/2$$

M. Zeng, L. Hu, H. Hu, Y. Z. You, C. Wu SCPMA 67, 237411 (2024)



g_{θ_-}/g_{ϕ_+}

	$\theta_- = \theta_1 - \theta_2$	ϕ_-
ϕ_+	TR-breaking normal phase	usual normal
$\theta_+ = \frac{\theta_1+\theta_2}{2}$	2-band SC	quartetting

Example

- $\Delta_{p_x} + i\Delta_{p_y}$ chiral superconductor/superfluidity

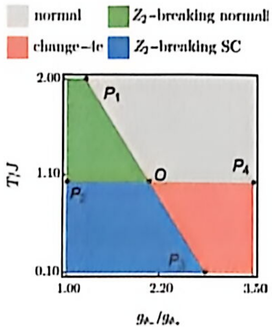
  Pairing of $L_z = 1$ with phase coherence

- $i(\Delta^+_{p_x}\Delta_{p_y} - h.c.)$ → chiral metal above Tc

  Pairing with $L_z = 1$ without phase coherence

- $\Delta^+_{p_x}\Delta^+_{p_y}$ → quartetting superconductivity

  TR symmetry preserved



M. Zeng, L. Hu, H. Hu, Y. Z. You, CW, SCPMA. 67, 237411 (2024).

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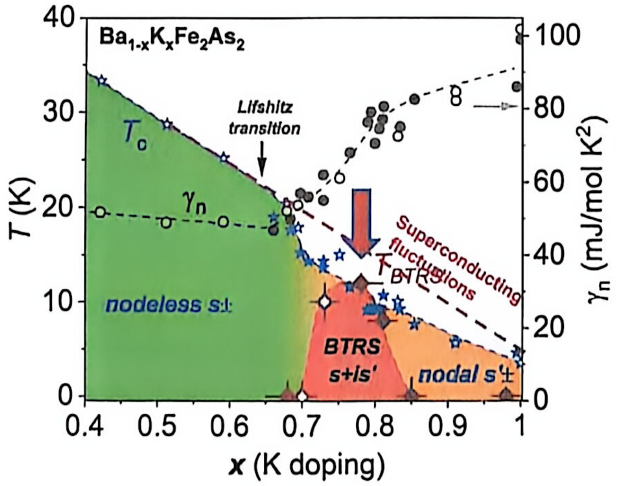
$Ba_{1-x}K_xFe_2As_2$: time-reversal breaking above Tc



V. Grinenko et al., Phys. Rev. B 95, 214511 (2017).; V. Grinenko et al., *Nat. Phys.* 16, 789 - 794 (2020).

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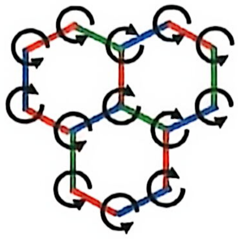
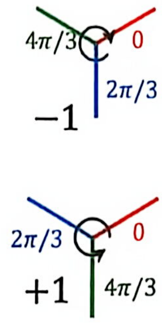
3-coloring (Baxter 1970) → sextetting

- Bond color (Z_3 variable)

6 coloring patterns → two vorticities



$$Z_N \approx 3^{\frac{3}{2}N}\left(\frac{6}{27}\right)^N$$



Pauling entropy: $S \approx Nk_B \ln W,\ W \approx 1.15$

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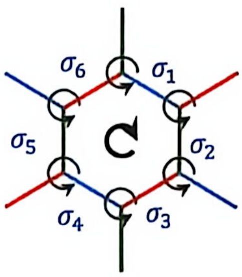
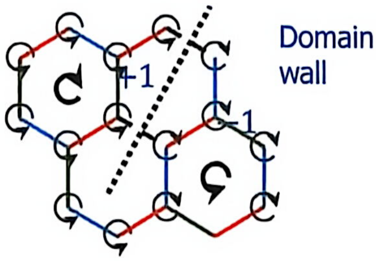
Fractional vortices





- Color constraint satisfied:

$$\frac{\Phi}{2\pi} = \frac{1}{3}\sum \sigma_a = 0, \pm 2\,,$$

- Fractional vortices

J. Moore and D. H. Lee, PRB, 2004.

(4+, 2-) mod 1 → -1/3

- External flux $\frac{\phi}{2\pi} = \frac{N}{3}$ offset by fractional vortices → charge 6e periodicity oscillation

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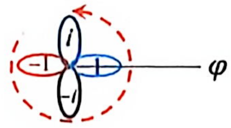
Vector potential

- Onsite U(1) phase and Ising variable.

φ: the phase of the right lobe.

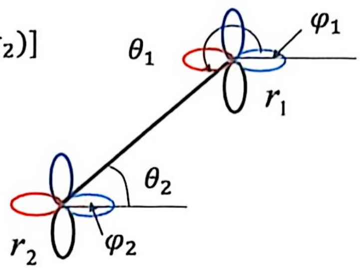
τ: direction of L_z.



- Inter-site Josephson coupling.

$$H_{eff} = -J_{||}\Sigma\cos\left[\varphi_1 - \varphi_2 + A_{||}(\tau_1, \tau_2)\right]$$

$$A_{||}(\tau_1, \tau_2) = \tau_1\theta_1 - \tau_2\theta_2$$



J. Moore and D. H. Lee, PRB, 2004.

C. Wu, Mod. Phys. Lett. 23, 1(2009).

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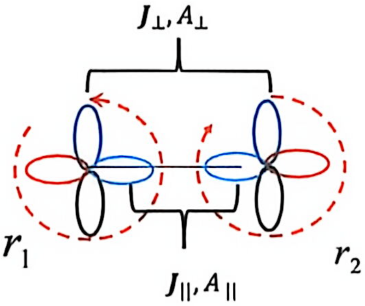
Vector potential

$$H_{eff} = -J_{||}\Sigma\cos\left[\varphi_1 - \varphi_2 + A_{||}(\tau_1, \tau_2)\right]$$
$$-J_{\perp}\,\Sigma\cos\left[\varphi_1 - \varphi_2 + A_{\perp}(\tau_1, \tau_2)\right]$$

- $J_{||}$ -- phase coherence along the longitudinal direction

- $J_{\perp}$ -- phase coherence along the transverse direction

$J_{\perp} \ll J_{||}$ → opposite chirality



J. Moore and D. H. Lee, PRB, 2004. C. Wu, Mod. Phys. Lett. 23, 1(2009).

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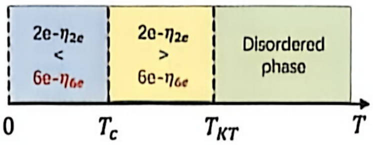
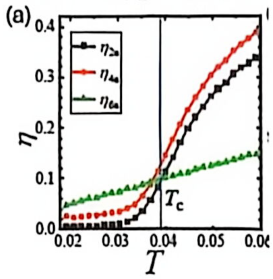
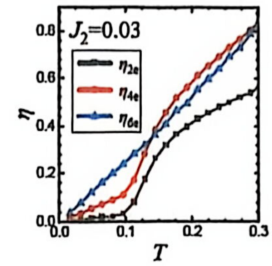
Decay powers of 2e, 4e, and 6e correlations

$$H = H_0 + H_2 = -J\sum_{\langle i,j\rangle}\cos\left[\theta_i-\theta_j+A_\parallel\right] - J_2\sum_{\langle i,j\rangle}\cos\left[\theta_i-\theta_j+A_\perp\right]$$



$J = 1, J_2 = 0.01$





Z. Pan, C. Lu, F. Yang, **CW**, SCPAM. 67, 287412 (2024).

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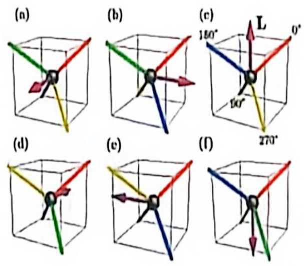
4-boson superfluidity – diamond lattice

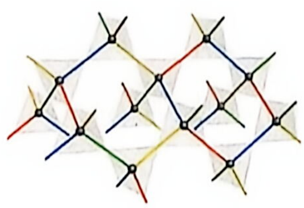
- P-orbital BEC with cubic anisotropy.

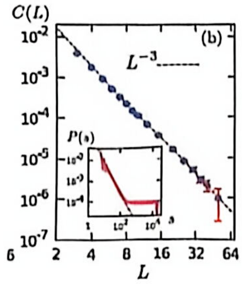
  4-coloring model
  R, G, B, Y: $0, \pm\frac{\pi}{2}, \pi$

  $4! = 24, \quad \vec{L} \,||\, \pm\hat{x}, \pm\hat{y}, \pm\hat{z}$

- Coulomb phase → dipolar correlation $\frac{1}{r^3}$







G. W. Chern, CW, PRL 112, 020601 (2011).

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Condensation at non-zero momenta

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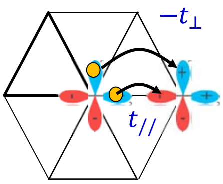

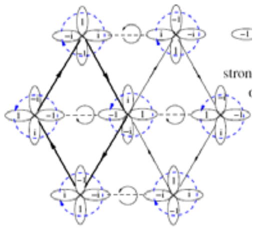

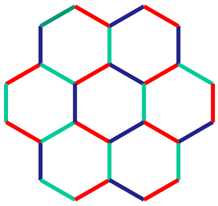

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1

U(1) phase coherence – analogy to FM

- Superconductivity → FM XY model

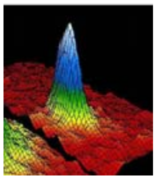
$$H = -J \sum_{ij} \cos(\theta_i - \theta_j)$$

- "No-node" theorem: Ground state wavefunctions of bosons

superfluid ^{4}He, cold alkali atom BEC, etc

$$\psi(r_1, r_2, \dots r_n) \geq 0$$

Feynman's textbook "*Statistical Mechanics*"



- **frustration → condensations at non-zero momenta**

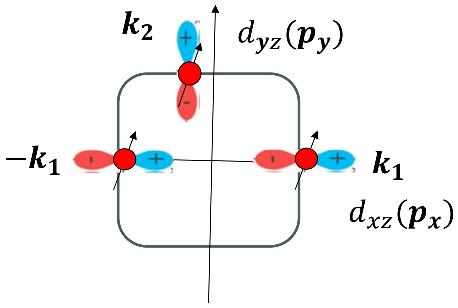
2

Cond. at nonzero momenta(I) — Inter-orbital pairing

- Triplet pairing in multi-orbital systems.

  intra-orbital pairing unfavorable by Pauli's exclusion

  inter-orbital pairing → Cooper pairing at non-zero momenta



3

Microscopic model — spinless 2-orbital Hubbard

p_x, p_y $\quad H_{int} = -U \sum_i n_{px}(i) n_{py}(i)$

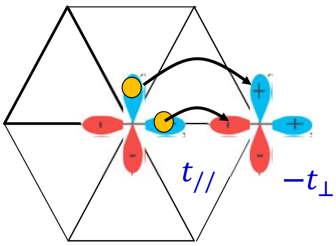
- Pseudo-spin SU(2) —— Yang and Zhang

$\eta_+ = p_x^+ p_y^+, \qquad \eta_- = p_y p_x,$

$\eta_z = \frac{1}{2}(n-1)$

opposite sign

σ and π-bondings



- Hopping along x-direction — opposite signs of σ and π bonding

$$H_i = t_{||} \sum_i p_x^+(i) p_x(i+\hat{x}) - t_{\perp} \sum_i p_y^+(i) p_y(i+\hat{x}) + \cdots$$

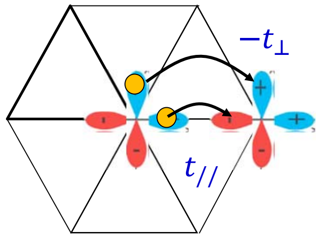
4

Frustrated Cooper pair hopping

- Pairing hopping

$$\langle G|\eta_x(i)|G\rangle = Re\Delta_i,$$

$$\langle G|\eta_y|G\rangle = Im\Delta_i,\ \langle G|\eta_z|G\rangle = \tfrac{1}{2}(n\text{-}1)$$



π-Josephson coupling

$$J_{xy} = \frac{4t_\perp t_{||}}{U} > 0 \qquad J_z = \frac{2(t_\perp^2 + t_{||}^2)}{U}$$

$$H_{eff} = \frac{J_{xy}}{2}\sum\big(\eta_+(i)\eta_-(j) + \eta_-(i)\eta_+(j)\big) + J_z\sum\eta_z(i)\eta_z(j) - \mu\sum\eta_z(i)$$

Zeeman-field

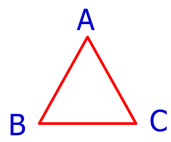
H.H. Hung, W. C.Lee, **CW**, PRB 83, 144506 (2011)

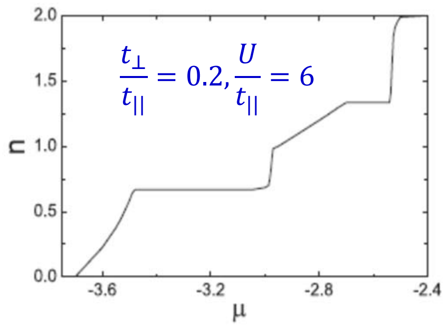
5

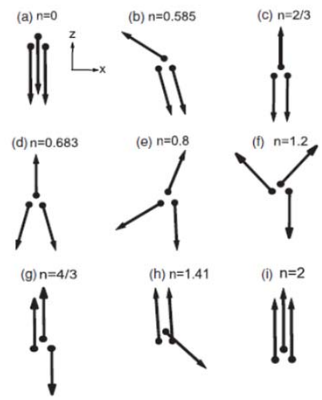
Umbrella configuration of CDW and gap function

- BCS results for the Hubbard model

  a $\sqrt{3}\times\sqrt{3}$ supercell







6

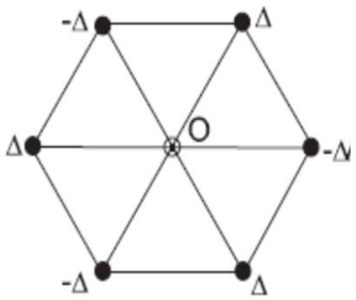
Spatial sign-change gap function

$$\Delta(\vec{r}) = \sin \vec{G}_1 \cdot \vec{r} + \sin \vec{G}_2 \cdot \vec{r} + sin\vec{G}_3 \cdot \vec{r}$$

$$\vec{G}_1 = (\frac{4\pi}{3a}, 0)$$

$$\vec{G}_2 = (-\frac{2\pi}{3a}, \frac{2\pi}{\sqrt{3}a})$$

$$\vec{G}_3 = (-\frac{2\pi}{3a}, -\frac{2\pi}{\sqrt{3}a})$$



$t_\perp = 0.2, t_{||} = 1, U = 6$

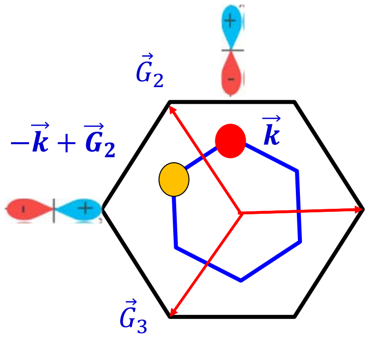
H.H. Hung, W. C.Lee, **CW**, PRB 83, 144506 (2011)

$n \approx 1.07$: $n_+ = n_- = 1.54$, $n_O = 0.15$

7

Weak-coupling picture

Pairing between p_x, p_y patches on Fermi surfaces
→ non-zero momentum pairing



The reduced BZ and $\sqrt{3} \times \sqrt{3}$ supercell

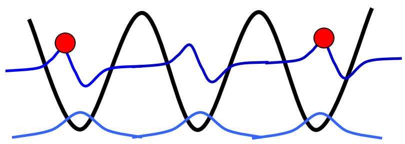
$$\vec{G}_1 = (\frac{4\pi}{3a}, 0)$$

8

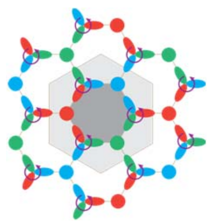
(II) Complex BECs at non-zero momenta in high-orbital bands

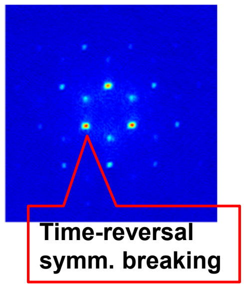
Meta-stable excited states

- Unconstrained by "no-node"
- cond. at non-zero momenta
- Complex-valued WF and time-reversal symmetry breaking



C. Wu, Mod. Phys. Lett. 23, 1 (2009) (brief review). W. V. Liu and C. Wu, PRA 2006





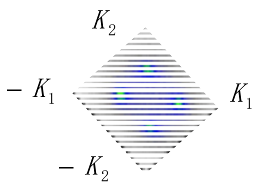
Experiment: Hemmerich's group Nature Physics 7, 147 (2011); PRL 114, 115301 (2015).

Xiao-Qiong Wang et al., Nature 596, 227 (2021)

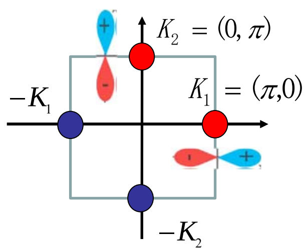
Wang, Luo, Liu, Huang, Li, **CW**, A. Hemmerich, Z. F. Xu, PRL 131, 226001 (2023).

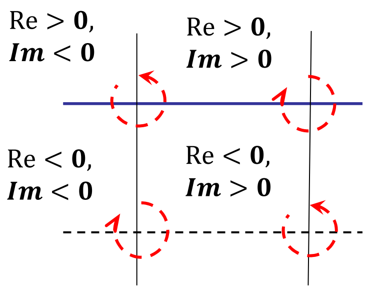
9

Square lattice: spontaneous TR breaking



$$\Psi(\vec{r}) = \Psi_{K1}(\vec{r}) + i\Psi_{K2}(\vec{r})$$





degenerate minima

complex supposition → nodal points → energetically favorable

Vortex-antivortex config.

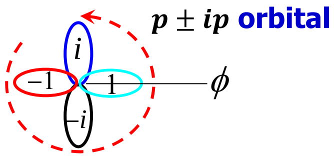
10

Vortex coupling — lattice model

- Onsite U(1) phase and Ising variable.

ϕ : the phase of the right lobe.

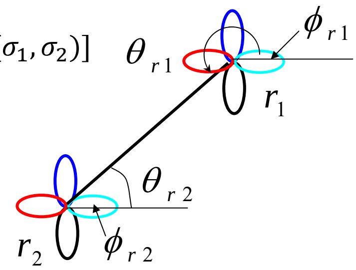
σ : direction of the Lz.



- Inter-site Josephson coupling.

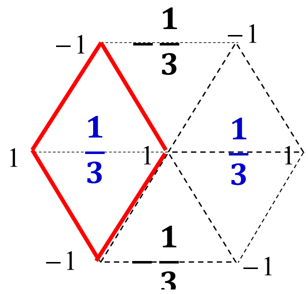
$$H_{eff} = -\frac{nt_{||}}{2} \sum \cos\left[\phi_{r1} - \phi_{r2} + A_{r1,r_2}(\sigma_1, \sigma_2)\right]$$

$$A_{r1,r2}(\sigma_{r1}, \sigma_{r2}) = \sigma_{r1}\theta_{r1} - \sigma_{r2}\theta_{r2}$$



J. Moore and D. H. Lee, PRB, 2004.

P-orbital BEC in a triangular lattice

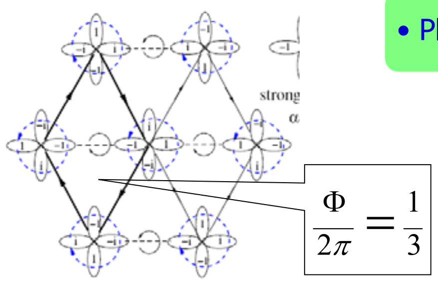


- Minimal vorticity/plaquette $\pm 1/6$

$$\Phi_i = \frac{1}{2\pi} \sum_{\langle r,r' \rangle} A_{r,r'} = \frac{1}{6}(\sigma_{r1} + \sigma_{r2} + \sigma_{r3})$$

- The stripe pattern minimizes vorticity

- Plaquette loop current



CW, W. Vincent Liu, J. Moore, and S. Das Sarma , PRL 97, 190406 (2006)

Experimental: Wang, Luo, Liu, Huang, Li, **CW**, A. Hemmerich, Z. F. Xu, PRL 131, 226001 (2023).

Hund's coupling assisted high Tc superconductivity in $\mathrm{La_3Ni_2O_7}$

Congjun Wu

Department of Physics, Westlake University

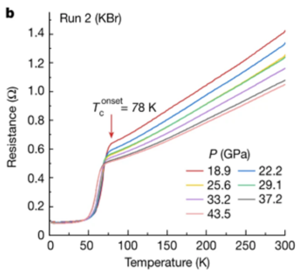
Collaborators

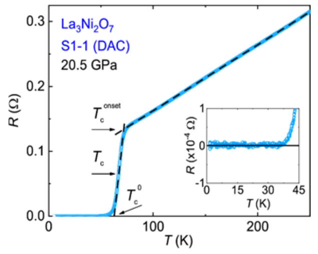
Chen Lu, Zhiming Pan	(Westlake University)
Fan Yang	(Beijing Institute of Technology)
Wei Li, Gang Su	(ITP, CAS)
Tianxi Ma, Runyu Ma	(Beijing Normal University)

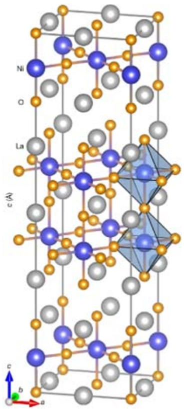
1. Chen Lu, Zhiming Pan, Fan Yang, Congjun Wu, Phys. Rev. Lett. 132, 146002 (2024).
2. Xing-Zhou Qu, Dai-Wei Qu, Jialin Chen, Congjun Wu, Fan Yang, Wei Li, Gang Su, Phys. Rev. Lett. 132, 036502 (2024).
3. Tianxing Ma, Da Wang, Congjun Wu, "Doping-driven Antiferromagnetic Insulator - Superconductor Transition: a Quantum Monte Carlo Study", Phys. Rev. B 106, 054510(2022).

Bilayer Ruddlesden–Popper $La_3Ni_2O_7$

signatures of superconductivity near 80 K under high pressure







Zero resistivity

Meng Wang, G. M. Zhang et al., Nature 621, 493 (2023)

Huiqiu Yuan et al., arxiv:2307.14819 (2023)

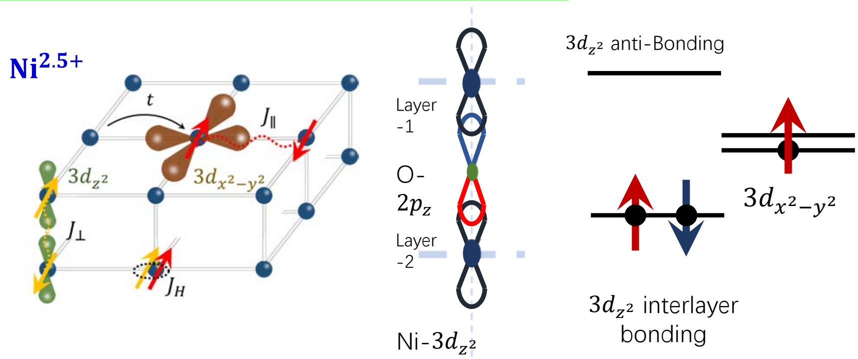
Z. Luo, X. Hu, M. Wang, W. Wu and D. X. Yao, PRL 131.126001(2023)

Many theoretical works………(sorry for no time to mention)

3

Relevant degrees of freedom

- Bilayer two-orbitals $3d_{x^2-y^2}$ and $3d_{z^2}$



Ni-$3d_{z^2}$ interlayer hopping by O-$2p_z$

→ Interlayer σ-bonding

Three e_g electrons for the two Ni ions

4

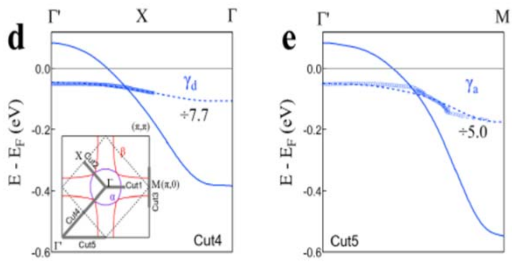
Dilemma

- $d_{x^2-y^2}$ is heavily overdoped, but its inter-layer tunneling is small

- d_{z^2}- is nearly half-filled, but its in-plane bandwith is small

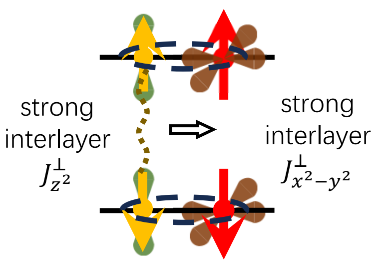
- Neither is good for high Tc .

Xinjiang Zhou's group, 2023.01148



Rescue from Hund's coupling

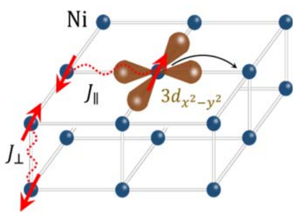
- Hund's coupling: spin aligned in partially filled orbitals.



- interlayer $J^{\perp}_{z^2}$ in $3d_{z^2}$

→

interlayer $J^{\perp}_{x^2-y^2}$ in $3d_{x^2-y^2}$

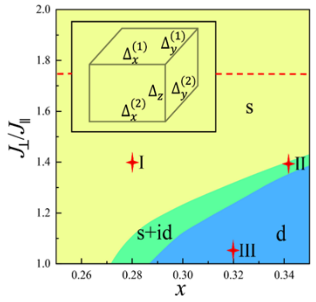


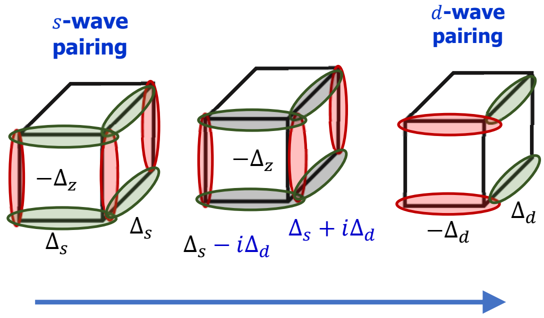
- A minimal model:

bilayer t-$J_{\perp}$-$J_{\parallel}$ model of $3d_{x^2-y^2}$

C. Lu, ZM Pan, F. Yang, C. Wu, Phys. Rev. Lett. 132, 146002 (2024).

Pairing Symmetry





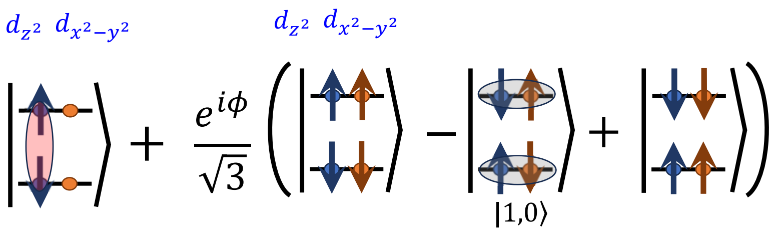
Filling $3d_{x^2-y^2}$ orbital

C. Lu, ZM Pan, F. Yang, C. Wu, Phys. Rev. Lett. 132, 146002 (2024).

- **Decrease $J_\perp/J_{||}$: from s-wave to d-wave**
- **Spontaneously time-reversal symmetry breaking $s+id$ phase**

7

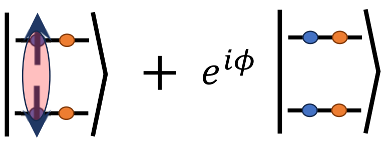
Hund's rule: $d_{x^2-y^2}$ interlayer pairing



spin-$\frac{1}{2}$ rung singlet

spin-1 rung singlet

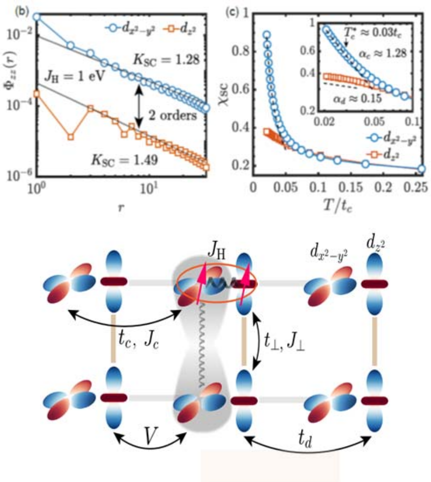
d_{z^2} interlayer singlet pairing



8

Confirmation from the two-orbital *t-J-J_H* model

- $d_{x^2-y^2}$ interlayer pairing dominate



$U = 4$ eV

$d_{x^2-y^2}: t_c = 0.483$ eV,

$J_c \simeq 4t_c^2/U = 0.233$ eV

$\varepsilon_c = 0.776$ eV,

$d_{z^2}: \; t_d = 0.110$ eV,

$t_\perp = 0.635$ eV,

$J_\perp \simeq 4t_\perp^2/U = 0.403$ eV.

$\varepsilon_d = 0.409$ eV,

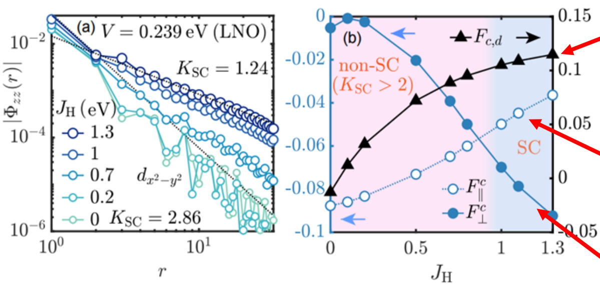
$d_{x^2-y^2} \leftrightarrow d_{z^2} \quad V = 0.239$ eV,

X.-Z. Qu, D.-W. Qu, W. Li, and G. Su, arxiv: 2311.12769 (2023)

9

Relevancy of Hund's coupling

- SC appears when J_H reaches 1eV



onsite interorbital FM correlation

Intra-layer $d_{x^2-y^2}$ AFM correlation

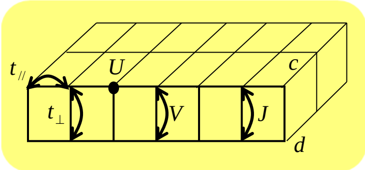
Inter-layer $d_{x^2-y^2}$ AFM correlation

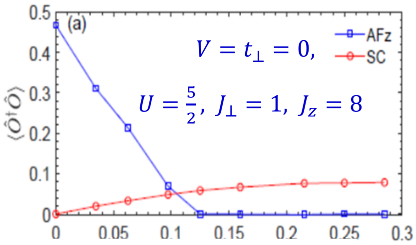
X.-Z. Qu, D.-W. Qu, W. Li, and G. Su, arxiv: 2311.12769 (2023)

10

Bilayer extended Hubbard: Dope AFM → SC (QMC)

T. Ma, D. Wang, **CW**, PRB 106, 054510(2022), arxiv 1806.03652.

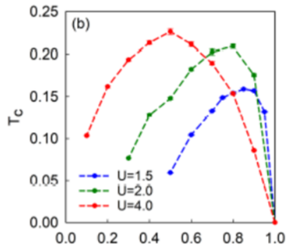




- Kramers symm. decomp.

CW, S C. Zhang PRB 71, 155115 (2005).

- AFM destabilized and SC developing with doping
- Interlayer s-wave.



$V = t_\perp = 0,$

$J_\perp = J_z = \frac{4}{3}U$

$\langle n \rangle = 1 - x$

Summary $d_{x^2-y^2}$ is the driving force

1) SC arises from dx2-y2 orbital, not directly from dz2-orbital. Hence, it is ok if the dz2-orbital does not cross the Fermi surface.
2) The most possible pairing symmetry is the inter-layer s-wave one based on dx2-y2 orbital, not the intra-layer d-wave one.
3) The interlayer pairing between dx2-y2 orbitalis assisted by the onsite Hund's coupling between dx2-y2 and dz2-orbitals and the inter-layer AFM exchange via the dz2-orbital.