Covariant derivative in Riemann space

Consider a space with a coordinate system S in which a point is represented as $P:(x^1, \cdots x^n)$. The coordinates are independant, i.e $\partial x^i/\partial x^j = \delta^i_j$. We can use another coordinate system $\bar{S}$ in which P is represented as $(\bar{x}^1, \cdots \bar{x}^n)$. The two sets of coordinates are related by

$$\boxed{\bar{x}^i = \bar{x}^i(x^1, \cdots x^n), \quad x^i = x^i(\bar{x}^1, \cdots \bar{x}^n)}.$$

Affine space: we define the derivative

$d\bar{x}^i = \dfrac{\partial \bar{x}^i}{\partial x^j} dx^j$, then $\boxed{\{dx^j\}}$ can be viewed as a componet of a contravariant vector. Consider a covariant vector, we consider the gradient of a scalar function

$$\phi(x^1, \cdots x^n) = \phi(\bar{x}^1, \cdots \bar{x}^n) \Rightarrow \frac{\partial \phi}{\partial x^i} dx^i = \frac{\partial \phi}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i} d\bar{x}^i = \frac{\partial \phi}{\partial \bar{x}^i} d\bar{x}^i$$

$$\Rightarrow \frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i} \Rightarrow \left\{\frac{\partial \phi}{\partial \bar{x}^i}\right\}$$ a component of a covariant vector.

In an affine space, the covariant and contravariant vector/tensors are unrelated. (Manifold where parallel transport of vectors is defined is called affine space. — Metric is not needed!)

Metric space: In a metric space, we define the metric tensor $g_{ij}(x^1, \cdots x^n)$ to relate covariant and contravariant vectors

$$v_i = g_{ij} v^j, \quad \text{and} \quad v^i = g^{ij} v_j \rightarrow (g_{ij})^{-1} = g^{ij}$$

(the metric tensor is symmetric).

$$v^2 = v_i v^i = g_{ij} v^i v^j = g^{ij} v_i v_j$$

→ the arc length $ds^2 = g_{ij}\, dx^i dx^j$.

If the metric g_{ij} is always positive-definite, then such a space is called Riemann space.

Ex: 2D sphere $ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta\, d\varphi^2$ — 2D Rienann space.

⊛ Parallel transport in a curved manifold

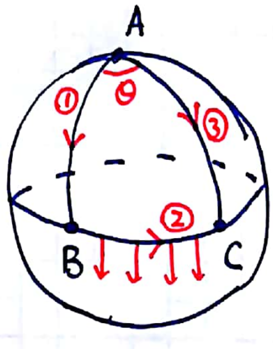
The derivative of a vecter is typically defined as

$$\partial_\mu A^\nu(x) = \lim \frac{A^\nu(x+\epsilon\hat{n}) - A^\nu(x)}{\epsilon}$$

, where $\hat{n}$ is the unit vector along the direction of x^μ. However, two vecters at two different places cannot be compared directly. For example, at two different points, the tangent planes are different, and the tangent vectors are not in the same plane. In order to compare them, we need to parallel transport $A^\nu(x)$ to the position of $x+\epsilon\hat{n}$, and then take the comparison.

Example of parallel transport: For convenience, we use the unit vectors in the spherical coordinate system: $\hat{e}_\theta$, $\hat{e}_\varphi$

$$\vec{B} = B_\theta \hat{e}_\theta + B_\varphi \hat{e}_\varphi$$



How to understand parallel ~~transport~~: $\vec{B}$ is impossible to be strictly parrell in the 3D sense. otherwise, $\vec{B}$ will not lie in the tangent plane. But

③

we should restrict no rotation in the tangent plane, i.e. $d\vec{B} \parallel \hat{e}_r$.

$$d\vec{B}\cdot\hat{e}_\theta = d\vec{B}\cdot\hat{e}_\varphi = 0 .$$

$$d\vec{B} = dB_\theta\,\hat{e}_\theta + dB_\varphi\,\hat{e}_\varphi + A_\theta\, d\hat{e}_\theta + A_\varphi\, d\hat{e}_\varphi$$

$$d\hat{e}_\theta = -d\theta\,\hat{e}_r + \cos\theta\, d\varphi\,\hat{e}_\varphi ,\quad d\hat{e}_\varphi = -\sin\theta\, d\varphi\,\hat{e}_r - \cos\theta\, d\varphi\,\hat{e}_\theta$$

$$\Rightarrow d\vec{B} = (-B_\theta d\theta - B_\varphi \sin\theta\, d\phi)\hat{e}_r + (dB_\theta - B_\varphi\cos\theta\, d\phi)e_\theta$$
$$+(dB_\varphi + A_\theta\cos\theta\, d\varphi)\,\hat{e}_\varphi$$

$$\boxed{\begin{cases} dB_\theta = B_\varphi \cos\theta\, d\phi \\ dB_\varphi = -B_\theta\cos\theta\, d\theta \end{cases} \rightarrow d(B_\theta + iB_\varphi) = -i(B_\theta + iB_\varphi)\cos\theta\, d\varphi}$$

For example, a vector $\vec{B}$ parallelly transport from $A\to B \to C \to A$. At the end of transport, it is rotated at the angle of Θ. — the solid angle spanned by the triangle of $\widehat{ABC}$. But (B_θ, B_φ) are not the coordinates in the sense of vector. In order to follow the usual coordinate transformation.

$$\vec{r}_\theta = \frac{\partial \hat{e}_r}{\partial\theta} = \hat{e}_\theta ,\quad \vec{r}_\varphi = \frac{\partial \hat{e}_r}{\partial\varphi} = \sin\theta\,\hat{e}_\varphi \leftarrow \boxed{d\hat{e}_r = d\theta\,\hat{e}_\theta + \sin\theta\, d\varphi\,\hat{e}_\varphi}$$

$$\vec{A} = A^\theta\,\vec{r}_\theta + A^\varphi\,\vec{r}_\varphi ,$$

$$d\vec{A} = dA^\theta\,\vec{r}_\theta + dA^\varphi\,\vec{r}_\varphi + A^\theta \frac{\partial\hat{e}_\theta}{\partial\varphi} d\varphi + A^\varphi\left(\frac{\partial\vec{r}_\varphi}{\partial\theta}d\theta + \frac{\partial\vec{r}_\varphi}{\partial\varphi}d\varphi\right)$$

$$= dA^\theta\,\vec{r}_\theta + dA^\varphi\,\vec{r}_\varphi + A^\theta\frac{\cos\theta}{\sin\theta}d\varphi\,\vec{r}_\varphi + A^\varphi\left(\frac{\cos\theta}{\sin\theta}d\theta\,\vec{r}_\varphi - \sin\theta\cos\theta\, d\varphi\,\vec{r}_\theta\right)$$

$$\boxed{\begin{aligned} d\vec{A} &= \left[dA^\theta - \sin\theta\cos\theta\, A^\varphi\, d\varphi\right]\vec{r}_\theta \\ &+ \left[dA^\varphi + \cot\theta\,(A^\theta d\varphi + A^\varphi d\theta)\right]\vec{r}_\varphi \end{aligned}} \leftarrow \text{drop the change along } \hat{e}_r .$$

Hence, the parallel transport means

$$dA^\theta = \sin\theta\cos\theta\, A^\varphi d\varphi$$
$$dA^\varphi = -\cot\theta\,[A^\theta d\varphi + A^\varphi d\theta]$$

$$A^{\mu,*}(x+dx) = A^\mu(x) - \Gamma^\mu_{\nu\lambda} dx^\nu A^\lambda(x)$$

$$\boxed{\frac{\partial \vec{r}_\nu}{\partial\lambda} = \frac{\partial\vec{r}_\lambda}{\partial\nu} \Rightarrow \Gamma^\mu_{\nu\lambda} \text{ are symmetric under } \nu\leftrightarrow\lambda!}$$

$$\Rightarrow \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \cot\theta.$$

The $\Gamma^\mu_{\nu\lambda}$ defined above is called Christoffle symbol, which reflects ~~the tangent vectors~~ variation of the tangent vectors. Compared to $\Gamma^\mu_{\nu\lambda}$, $(\mu,\nu\lambda = r,\theta,\varphi)$ for the spherical coordinate but for 3D flat space, we project out the variantions of $\vec{r}_\theta$, $\vec{r}_\varphi$ on the $\hat{e}_r$ direction, Only keep those projected in the tangent plane. This is the origin of curature.

⊛ Question: how $\Gamma^\mu_{\nu\lambda}$ transforms under coordinate transformation.

If we change a coordinate system

$$A^{*,\prime\mu}(x'+dx') = A'^\mu(x') - \Gamma'^\mu_{\nu\lambda} dx'^\nu A'^\lambda(x')$$

we require that parallel transport does not change the nature of a vector.

$$A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\lambda} A^\lambda(x)$$

$$A^{*\prime\mu}(x'+dx') = \left.\frac{\partial x'^\mu}{\partial x^\lambda}\right|_{x+dx} A^{*,\lambda}(x+dx)$$

$$\rightarrow \left.\frac{\partial x'^\mu}{\partial x^\lambda}\right|_{x+dx} A^{*,\lambda}(x+dx) = \frac{\partial x'^\mu}{\partial x^\lambda} A^\lambda(x) - \Gamma'^\mu_{\nu\lambda}(x) dx'^\nu A'^\lambda(x')$$

$$\rightarrow \left.\frac{\partial x'^\mu}{\partial x^\lambda}\right|_{x+dx} \left(A^\lambda(x) - \Gamma^\lambda_{\kappa\eta}(x)\, dx^\kappa A^\eta(x)\right) = \frac{\partial x'^\mu}{\partial x^\lambda} A^\lambda(x) - \Gamma'^\mu_{\nu\lambda}(x')\, dx'^\nu A'^\lambda(x')$$

expand $$\left.\frac{\partial x'^\mu}{\partial x^\lambda}\right|_{x+dx} = \left.\frac{\partial x'^\mu}{\partial x^\lambda}\right|_x + \frac{\partial^2 x'^\mu}{\partial x^\eta \partial x^\lambda} dx^\eta$$

Keep linear infinitesimal $\Rightarrow$

$$-\frac{\partial x'^\mu}{\partial x^\lambda}\Gamma^\lambda_{k\eta}(x)\, dx^k A^\eta(x) + \frac{\partial^2 x'^\mu}{\partial x^\eta \partial x^\lambda} dx^\eta A^\lambda(x) = -\Gamma'^\mu_{\nu\lambda}(x')\, dx'^\nu A'^\lambda(x')$$

$$-\frac{\partial x'^\mu}{\partial x^\lambda}\Gamma^\lambda_{k\eta}(x)\frac{\partial x^k}{\partial x'^\sigma} dx'^\sigma \frac{\partial x^\eta}{\partial x'^\nu} A'^\nu(x') + \frac{\partial^2 x'^\mu}{\partial x^\eta \partial x^\lambda}\frac{\partial x^\eta}{\partial x'^\sigma} dx'^\sigma \frac{\partial x^\lambda}{\partial x'^\nu} A'^\nu(x)$$

$$= -\Gamma'^\mu_{\sigma\nu}(x')\, dx'^\sigma A'^\nu(x')$$

$$\Rightarrow \boxed{\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x^k}{\partial x'^\sigma}\frac{\partial x^\eta}{\partial x'^\nu}\Gamma^\lambda_{k\eta}(x) - \frac{\partial^2 x'^\mu}{\partial x^\eta \partial x^\lambda}\frac{\partial x^\eta}{\partial x'^\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} = \Gamma'^\mu_{\sigma\nu}(x')}$$

$\Gamma^\mu_{\sigma\nu}$ is not a tensor because the coordinate transformations at x and $x+dx$ are different! This brings an extra term!

⊛ Christoffel symbol

For an affine space, we do not need to define a metric for the purpose of parallel transport. However, if we want the length is also invariant during the transport, we need to define metric, $g_{\mu\nu}(x)$. Based on $g_{\mu\nu}$, we can define $\Gamma^\mu_{\nu\lambda}(x)$.

Consider two vectors A^μ and V^μ, along a curve $x(\tau)$. The scalar product along the curve is

$$g_{\mu\nu}(x(\tau))\, A^\mu(x(\tau))\, V^\nu(x(\tau)),$$ and we want this scalar is invariant along $x(\tau)$, during parallel transport.

$$\frac{d}{d\tau}(g_{\mu\nu} A^\mu V^\nu) = 0 \Rightarrow \frac{d}{d\tau} g_{\mu\nu} A^\mu V^\nu + g_{\mu\nu}\frac{dA^\mu}{d\tau} V^\nu + g_{\mu\nu} A^\mu \frac{dV^\nu}{d\tau} = 0$$

For parallel transport, $\frac{dA^\mu}{d\tau} = -\Gamma^\mu_{\lambda\rho}\frac{dx^\lambda}{d\tau} A^\rho$, $\frac{dV^\nu}{d\tau} = -\Gamma^\nu_{\lambda\nu}\frac{dx^\lambda}{d\tau} V^\rho$

$$\Rightarrow \left[\partial_\lambda g_{\mu\nu} A^\mu V^\nu + g_{\mu\nu}\Gamma^\mu_{\lambda\rho} A^\rho V^\nu + g_{\mu\nu} A^\mu \Gamma^\nu_{\lambda\rho} V^\rho\right]\frac{dx^\lambda}{d\tau} = 0$$

$$\boxed{\left[\partial_\lambda g_{\mu\nu} + g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} + g_{\sigma\mu}\Gamma^\sigma_{\nu\lambda}\right]} A^\mu V^\nu \frac{dx^\lambda}{d\tau} = 0$$

$$\Rightarrow \begin{cases} \partial_\lambda g_{\mu\nu} + g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} + g_{\sigma\mu}\Gamma^\sigma_{\nu\lambda} = 0 \\ \partial_\mu g_{\nu\lambda} + g_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} + g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} = 0 \\ \partial_\nu g_{\lambda\mu} + g_{\sigma\mu}\Gamma^\sigma_{\nu\lambda} + g_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} = 0 \end{cases}$$ ← cyclically permute $\lambda\mu\nu$

$$\rightarrow \frac{1}{2}\left[\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu}\right] + g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} + g_{\sigma\mu}\Gamma^\sigma_{\nu\lambda} + g_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} = 0$$

$$\Rightarrow g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} = +\frac{1}{2}\left[\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}\right]$$

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\sigma\nu}\left[\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}\right]$$

$$\boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}\left[\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right]}$$

$$\Rightarrow \Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\lambda}\left[\cancel{\partial_\mu g_{\nu\lambda}} + \partial_\nu g_{\lambda\mu} - \cancel{\partial_\lambda g_{\mu\nu}}\right]$$

$$= \frac{1}{2}\,\mathrm{tr}\left[g^{-1}\partial_\nu g\right] = \frac{1}{2}\partial_\nu\,\mathrm{tr}\ln g = \frac{1}{2}\partial_\nu \ln\det g$$

← in the sense of matrix

$$= \partial_\nu \ln\sqrt{\det g} \triangleq \frac{1}{\sqrt{g}}\partial_\nu\sqrt{g}$$

← det of g

$$\boxed{\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{g}}\partial_\nu\sqrt{g}}$$

Now we check the transformation law of Christoffle symbol

$$\Gamma'^{\sigma}_{\mu\nu} = \frac{1}{2} g'^{\sigma\lambda}\left(\partial'_\mu g'_{\nu\lambda} + \partial'_\nu g'_{\lambda\mu} - \partial'_\lambda g'_{\mu\nu} \right)$$

Look at terms in the parentdesis:

$$\partial'_\mu g'_{\nu\lambda} + \partial'_\nu g'_{\lambda\mu} - \partial'_\lambda g'_{\mu\nu} = \partial'_\mu \left[\frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} g_{\nu_1\lambda_1} \right] + \partial'_\nu \left[\frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \frac{\partial x^{\mu_1}}{\partial x'^\mu} g_{\lambda_1\mu_1} \right]$$

$$- \partial'_\lambda \left[\frac{\partial x^{\mu_1}}{\partial x'^\mu} \frac{\partial x^{\nu_1}}{\partial x'^\nu} g_{\mu_1\nu_1} \right]$$

$$= \frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \partial'_\mu g_{\nu_1\lambda_1} + \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \frac{\partial x^{\mu_1}}{\partial x'^\mu} \partial'_\nu g_{\lambda_1\mu_1} - \frac{\partial x^{\mu_1}}{\partial x'^\mu} \frac{\partial x^{\nu_1}}{\partial x'^\nu} \partial'_\lambda g_{\mu_1\nu_1}$$

$$+ \left(\frac{\partial^2 x^{\nu_1}}{\partial x'^\mu \partial x'^\nu} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} + \frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial^2 x^{\lambda_1}}{\partial x'^\mu \partial x'^\lambda} \right) g_{\nu_1\lambda_1}$$

$$+ \left(\frac{\partial^2 x^{\lambda_1}}{\partial x'^\nu \partial x'^\lambda} \frac{\partial x^{\mu_1}}{\partial x'^\mu} + \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \frac{\partial^2 x^{\mu_1}}{\partial x'^\nu \partial x'^\mu} \right) g_{\lambda_1\mu_1}$$

$$- \left(\frac{\partial^2 x^{\mu_1}}{\partial x'^\lambda \partial x'^\mu} \frac{\partial x^{\nu_1}}{\partial x'^\nu} + \frac{\partial x^{\mu_1}}{\partial x'^\mu} \frac{\partial^2 x^{\nu_1}}{\partial x'^\lambda \partial x'^\nu} \right) g_{\mu_1\nu_1}$$

Check the last 3 terms:

$$\frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \left[\frac{\partial^2 x^{\nu_1}}{\partial x'^\mu \partial x'^\nu} g_{\nu_1\lambda_1} + \frac{\partial^2 x^{\mu_1}}{\partial x'^\nu \partial x'^\mu} g_{\lambda_1\mu_1} \right]$$

$$+ \frac{\partial x^{\nu_1}}{\partial x'^\nu} \left[\frac{\partial^2 x^{\lambda_1}}{\partial x'^\mu \partial x'^\lambda} g_{\nu_1\lambda_1} - \frac{\partial^2 x^{\mu_1}}{\partial x'^\lambda \partial x'^\mu} g_{\mu_1\nu_1} \right] \to 0$$

$$+ \frac{\partial x^{\mu_1}}{\partial x'^\mu} \left[\frac{\partial^2 x^{\lambda_1}}{\partial x'^\nu \partial x'^\lambda} g_{\lambda_1\mu_1} - \frac{\partial^2 x^{\nu_1}}{\partial x'^\lambda \partial x'^\nu} g_{\mu_1\nu_1} \right] \to 0$$

$$= 2 g_{\nu_1\lambda_1} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \frac{\partial^2 x^{\nu_1}}{\partial x'^\mu \partial x'^\nu}$$

The first line: $\frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \frac{\partial x^{\mu_1}}{\partial x'^\mu} \left(\partial_{\mu_1} g_{\nu_1\lambda_1} + \partial_{\nu_1} g_{\lambda_1\mu_1} - \partial_{\lambda_1} g_{\mu_1\nu_1} \right)$

And $g'^{\sigma\lambda} = \frac{\partial x'^\sigma}{\partial x^{\sigma_1}} \frac{\partial x'^\lambda}{\partial x^{\lambda_1}} g^{\sigma_1\lambda_1}$ $\Rightarrow$ Combine with the first line $\Rightarrow$ $\frac{\partial x'^\sigma}{\partial x^{\sigma_1}} \frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial x^{\mu_1}}{\partial x'^\mu} g^{\sigma_1\lambda_1} \left[\partial_{\mu_1} g_{\nu_1\lambda_1} + \partial_{\nu_1} g_{\lambda_1\mu_1} - \partial_{\lambda_1} g_{\mu_1\nu_1} \right]$

The other part

$$\delta^{\lambda_1}_{\lambda_2} \qquad \delta^{\sigma_1}_{\nu_1}$$

$$\frac{1}{\cancel{2}} \frac{\partial x'^\sigma}{\partial x^{\sigma_1}} \frac{\partial x'^\lambda}{\partial x^{\lambda_2}} g^{\sigma_1 \lambda_2} \cdot \cancel{2} g_{\nu_1 \lambda_1} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \frac{\partial^2 x^{\nu_1}}{\partial x'^\mu \partial x'^\nu}$$

$$= \frac{\partial x'^\sigma}{\partial x^{\sigma_1}} \frac{\partial^2 x^{\sigma_1}}{\partial x'^\mu \partial x'^\nu}$$

$$\Rightarrow \quad \Gamma'^\sigma_{\mu\nu} = \frac{\partial x'^\sigma}{\partial x^{\sigma_1}} \frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial x^{\mu_1}}{\partial x'^\mu} \Gamma^{\sigma_1}_{\mu_1 \nu_1} + \frac{\partial x'^\sigma}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu}$$

we had a previous transformation based on parallel transport

$$\Gamma'^\sigma_{\mu\nu} = \frac{\partial x'^\sigma}{\partial x^{\sigma_1}} \frac{\partial x^{\nu_1}}{\partial x'^\nu} \frac{\partial x^{\mu_1}}{\partial x'^\mu} \Gamma^{\sigma_1}_{\mu_1 \nu_1} - \frac{\partial^2 x'^\sigma}{\partial x^\eta \partial x^\lambda} \frac{\partial x^\eta}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu}$$

In fact, the inhomogenous terms are the same.

$$\frac{\partial x'^\sigma}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu} = \frac{\partial}{\partial x'^\mu}\left[\frac{\partial x'^\sigma}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial x'^\nu} \right] - \frac{\partial}{\partial x'^\mu}\left[\frac{\partial x'^\sigma}{\partial x^\lambda}\right] \frac{\partial x^\lambda}{\partial x'^\nu}$$

$$= \underbrace{\frac{\partial}{\partial x'^\mu} \delta^\sigma_\nu}_{0} - \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} \left[\frac{\partial x'^\sigma}{\partial x^\lambda} \right] \frac{\partial x^\lambda}{\partial x'^\nu}$$

$$= - \frac{\partial^2 x'^\sigma}{\partial x^\eta \partial x^\lambda} \frac{\partial x^\eta}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} \qquad \leftarrow \text{partial derivative}$$

Covariant derivative

For a scalar, its derivative transforms as a vector in coordinate transformation.

$$\phi(x) \to \phi'(x') = \phi(x),$$

$$\partial_\mu \phi \to \partial'_\mu \phi'(x') = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi(x),$$

$$\partial^\mu \phi \to \partial'^\mu \phi'(x') = \frac{\partial x'^\mu}{\partial x_\nu} \partial^\nu \phi(x).$$

However, a simple derivative on vector, does not work

$$A^\mu(x) \to A'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu(x)$$

$$\partial_\sigma A^\mu(x) \to \partial'_\sigma A'^\mu(x') = \frac{\partial x^\lambda}{\partial x'^\sigma} \partial_\lambda \left[\frac{\partial x'^\mu}{\partial x^\nu} A^\nu(x) \right]$$

$$= \frac{\partial x^\lambda}{\partial x'^\sigma} \frac{\partial x'^\mu}{\partial x^\nu} \partial_\lambda A^\nu(x) + \underline{\frac{\partial x^\lambda}{\partial x'^\sigma} \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\nu} A^\nu(x)}$$

~~Can~~ The 2nd term spoils the transformation of tensor, which is similarly to the case of $\Gamma^\mu_{\sigma\lambda}$. Remember

$$\Gamma^\mu_{\sigma\lambda}(x) \to \Gamma'^\mu_{\sigma\lambda}(x') = \frac{\partial x'^\mu}{\partial x^{\mu_1}} \frac{\partial x^{\sigma_1}}{\partial x'^\sigma} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \Gamma^{\mu_1}_{\sigma_1\lambda_1}(x) - \frac{\partial^2 x'^\mu}{\partial x^{\sigma_1} \partial x^{\lambda_1}} \frac{\partial x^{\sigma_1}}{\partial x'^\sigma} \frac{\partial x^{\lambda_1}}{\partial x'^\lambda}$$

Then

$$\Gamma^\mu_{\sigma\lambda} A^\lambda(x) \to \Gamma'^\mu_{\sigma\lambda}(x') A'^\lambda(x') = \frac{\partial x'^\mu}{\partial x^{\mu_1}} \frac{\partial x^{\sigma_1}}{\partial x'^\sigma} \Gamma^{\mu_1}_{\sigma_1\lambda}(x) A^\lambda(x)$$

$$- \frac{\partial^2 x'^\mu}{\partial x^{\sigma_1} \partial x^{\lambda_1}} \frac{\partial x^{\sigma_1}}{\partial x'^\sigma} \underbrace{\frac{\partial x^{\lambda_1}}{\partial x'^\lambda} \frac{\partial x'^\lambda}{\partial x^{\lambda_2}}}_{\delta^{\lambda_1}_{\lambda_2}} A^{\lambda_2}(x)$$

$$= \frac{\partial x'^\mu}{\partial x^{\mu_1}} \frac{\partial x^{\sigma_1}}{\partial x'^\sigma} \Gamma^{\mu_1}_{\sigma_1\lambda}(x) A^\lambda(x) - \underline{\frac{\partial^2 x'^\mu}{\partial x^{\sigma_1} \partial x^\nu} \frac{\partial x^{\sigma_1}}{\partial x'^\sigma} A^\nu(x)}$$

$$\boxed{D^{\sigma} A^{\mu} = \partial^{\sigma} A^{\mu} + g^{\sigma\rho}\, \Gamma^{\mu}_{\rho\lambda} A^{\lambda}}$$

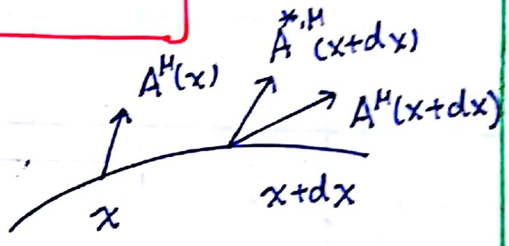
we define

$$\boxed{D_{\sigma} A^{\mu}(x) = \partial_{\sigma} A^{\mu}(x) + \Gamma^{\mu}_{\sigma\lambda}(x)\, A^{\lambda}(x)}$$

then

$$\boxed{D'_{\sigma} A'^{\mu}(x') = \frac{\partial x^{\sigma_1}}{\partial x'^{\sigma}} \frac{\partial x'^{\mu}}{\partial x^{\mu_1}} D_{\sigma_1} A^{\mu_1}(x)}$$

Recall the parallel transport



$$A^{*,\mu}(x+dx) = A^{\mu}(x) - \Gamma^{\mu}_{\sigma\lambda}\, dx^{\sigma} A^{\lambda}(x)$$

$$\boxed{\begin{aligned} A^{\mu}(x+dx) - A^{*,\mu}&(x+dx) \\ &\parallel \\ D_{\sigma} A^{\mu}(x)\, &dx^{\sigma} \end{aligned}} \begin{aligned} &= A^{\mu}(x+dx) - A^{\mu}(x) + \Gamma^{\mu}_{\sigma\lambda}(x) A^{\lambda}(x)\, dx^{\sigma} \\ \\ &= \left[\partial_{\sigma} A^{\mu}(x) + \Gamma^{\mu}_{\sigma\lambda}(x) A^{\lambda}(x)\right] dx^{\sigma} \end{aligned}$$

⊛ Covariant derivative on covariant component of vectors

For the covariant component $A_{\mu}(x)$, we also want its parallely transported counterpart $A^{*}_{\mu}(x+dx)$, transforms like a vector at $x+dx$.

i.e.
$$A^{*\prime}_{\mu}(x'+dx') = \left.\frac{\partial x^{\lambda}}{\partial x'^{\mu}}\right|_{x+dx} A^{*}_{\lambda}(x+dx)$$

we define
$$A^{*}_{\mu}(x+dx) = A_{\mu}(x) + \Gamma^{\lambda}_{\nu\mu}\, dx^{\nu} A_{\lambda}(x)$$

$$D_{\sigma} A_{\mu}(x)\, dx^{\sigma} = A_{\mu}(x+dx) - A^{*}_{\mu}(x+dx)$$

$$= A_{\mu}(x+dx) - A_{\mu}(x) - \Gamma^{\lambda}_{\sigma\mu}\, dx^{\sigma} A_{\lambda}$$

$$\Rightarrow \boxed{D_{\sigma} A_{\mu} = \partial_{\sigma} A_{\mu} - \Gamma^{\lambda}_{\sigma\mu} A_{\lambda}}$$

Similarly

$$\boxed{D^{\sigma} A_{\mu} = \partial^{\sigma} A_{\mu} - g^{\sigma\rho}\, \Gamma^{\lambda}_{\rho\mu} A_{\lambda}}$$

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Verification

$$D'_\sigma A'_\mu(x') = \partial'_\sigma A'_\mu(x') - \Gamma'^{\lambda}_{\sigma\mu}(x') A'_\lambda(x')$$

$$\frac{\partial x^{\lambda_2}}{\partial x'^{\lambda}} A_{\lambda_2}(x)$$

$$= \frac{\partial x^{\sigma_1}}{\partial x'^{\sigma}} \partial_{\sigma_1}\left[\frac{\partial x^{\mu_1}}{\partial x'^{\mu}} A_{\mu_1}(x)\right] - \left\{\frac{\partial x'^{\lambda}}{\partial x^{\lambda_1}} \frac{\partial x^{\sigma_1}}{\partial x'^{\sigma}} \frac{\partial x^{\mu_1}}{\partial x'^{\mu}} \Gamma^{\lambda_1}_{\sigma_1\mu_1}(x) + \frac{\partial x'^{\lambda}}{\partial x^{\lambda_1}} \frac{\partial^2 x^{\lambda_1}}{\partial x'^{\sigma}\partial x'^{\mu}}\right]$$

$$= \frac{\partial x^{\sigma_1}}{\partial x'^{\sigma}} \frac{\partial x^{\mu_1}}{\partial x'^{\mu}} \left[\partial_{\sigma_1} A_{\mu_1}(x) - \Gamma^{\lambda_1}_{\sigma_1\mu_1} A_\lambda(x)\right]$$

$$+ \underline{\frac{\partial x^{\sigma_1}}{\partial x'^{\sigma}} \frac{\partial^2 x^{\mu_1}}{\partial x'^{\sigma_2}\partial x'^{\mu}} \frac{\partial x'^{\sigma_2}}{\partial x^{\sigma_1}} A_{\mu_1} - \frac{\partial x^{\lambda_2}}{\partial x^{\lambda_1}} \frac{\partial^2 x^{\lambda_1}}{\partial x'^{\sigma}\partial x'^{\mu}} A_{\lambda_2}(x)}$$

$$A_\lambda$$

$$\delta^{\sigma_2}_{\sigma} \frac{\partial^2 x^{\mu_1}}{\partial x'^{\sigma_2}\partial x'^{\mu}} A_{\mu_1} - \delta^{\lambda_2}_{\lambda_1} \frac{\partial^2 x^{\lambda_1}}{\partial x'^{\sigma}\partial x'^{\mu}} A_{\lambda_2} = \left(\frac{\partial^2 x^{\lambda}}{\partial x'^{\sigma}\partial x'^{\mu}} - \frac{\partial^2 x^{\lambda}}{\partial x'^{\sigma}\partial x'^{\mu}}\right)$$

$$= 0$$

$$\Rightarrow \boxed{D'_\sigma A'_\mu(x') = \frac{\partial x^{\sigma_1}}{\partial x'^{\sigma}} \frac{\partial x^{\mu_1}}{\partial x'^{\mu}} D_{\sigma_1} A_{\mu_1}(x)}$$

we can also generalize the definition of covariant derivative to tensors

$$D_\sigma T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m} = \partial_\sigma T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m} + \sum_k \Gamma^{\mu_k}_{\sigma\lambda} T^{\mu_1\cdots\mu_{k-1}\lambda\mu_{k+1}\cdots\mu_n}_{\nu_1\cdots\nu_m}$$

$$- \sum_k \Gamma^{\lambda}_{\sigma\nu_k} T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_{k-1}\lambda\nu_{k+1}\cdots\nu_m}$$

$$D^\sigma T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m} = g^{\sigma\rho} D_\rho T^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_m}$$

Leibniz rule

$$D_\sigma(A^\mu B^\nu) = \partial_\sigma(A^\mu B^\nu) + \Gamma^{\mu}_{\sigma\lambda} A^\lambda B^\nu + \Gamma^{\nu}_{\sigma\lambda} A^\mu B^\lambda$$

$$= (\partial_\sigma A^\mu + \Gamma^{\mu}_{\sigma\lambda}) B^\nu + A^\mu(\partial_\sigma B^\nu + \Gamma^{\nu}_{\sigma\lambda} B^\lambda)$$

$$\Rightarrow D_\sigma(A^\mu B^\nu) = (D_\sigma A^\mu) B^\nu + A^\mu (D_\sigma B^\nu)$$

Let us consider the covariant derivative on the metric tensor

$$A_\mu(x) = g_{\mu\nu}(x) A^\nu(x)$$

$$\underbrace{D_\sigma A_\mu}_{①} = \underbrace{D_\sigma(g_{\mu\nu}) A^\nu}_{②} + \underbrace{g_{\mu\nu} D_\sigma A^\nu}_{③}$$

Since ① and ③ transform the same, we expect ② = 0

$$D_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} g_{\mu\lambda} = 0 \Rightarrow \boxed{D_\sigma g_{\mu\nu} = 0}$$

consistent with the previous results on Christoffle symbols.

Similarly, we also expect $\boxed{D_\lambda g^{\mu\nu} = 0}$, which is also the case.

$g^{\mu\nu} g_{\rho\nu} = \delta^\mu{}_\rho$ is a constand scalar

$$D_\lambda(g^{\mu\nu} g_{\rho\nu}) = g^{\mu\nu} D_\lambda g_{\rho\nu} + D_\lambda g^{\mu\nu} g_{\rho\nu} = D_\lambda g^{\mu\nu} g_{\rho\nu} = 0$$

Since $g_{\rho\nu}$ is arbitary, $D_\lambda g^{\mu\nu} = 0$.

$$\Rightarrow \partial_\lambda g^{\mu\nu} + \Gamma^\mu_{\lambda\sigma} g^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma} g^{\mu\sigma} = 0.$$

Similarly, $D^\lambda g^{\mu\nu} = D^\lambda g_{\mu\nu} = 0$. Hence, the covariant derivatives of the metric tensor is zero. This is called the metric compatibility condition.

> Metric determins parallel transport in such a way that itself is invariant under parallel transport.